Integer Eigenvectors and Subeigenvectors in Min-Plus Algebra

Nurul Fu'adi[1*], **Siswanto**[2], **Supriyadi Wibowo**[3], **Dwi Agustin Retnowardani**[4]
[1,2,3] Department of Mathematics, Universitas Sebelas Maret, Jl. Ir. Sutami No. 36A, Surakarta 57126, Indonesia
[4] Department of Statistics, Universitas PGRI Argopuro Jember, Jl. Jawa No. 10, Jember 68121, Indonesia

Corresponding Author.
**Email*: nurul25fuadi@student.uns.ac.id

Abstract: *We study integer eigenvectors and subeigenvectors in min-plus algebra by adapting theoretical results from max-plus algebra. Min-plus algebra is a commutative semiring with addition* $a \oplus' b = min(a,b)$ *and multiplication* $a \otimes' b = a + b$*. An eigenvector of a matrix* A *with respect to a scalar* λ *is a vector* x *satisfying the equation* $A \otimes' x = \lambda \otimes' x$*. Similarly, a subeigenvector of* A *with respect to* λ *is a vector* x *satisfying the inequality* $A \otimes' x \geq \lambda \otimes' x$*. In this paper, we focus on solutions where these vectors consist entirely of integers. We first provide the exact mathematical criteria for integer subeigenvectors to exist. We show that this existence depends on the principal eigenvalue of a scaled and rounded matrix, which is determined by the minimum average circuit weight of its associated graph. If the condition is satisfied, we can explicitly describe the complete set of these subeigenvectors. Secondly, we establish that an integer eigenvector can only correspond to the principal eigenvalue of the matrix. Finally, we demonstrate that determining an integer eigenvector is equivalent to finding an integer image in a column space, which can be solved using an iterative algorithm.*

Keywords: *Min-plus Algebra, Eigenvectors, Subeigenvectors, Integer, Column Space*

1. INTRODUCTION

Max-plus algebra is the foundation for modeling discrete event systems, with system stability represented by the eigen equation $A \otimes x = \lambda \otimes x$ [1]. The versatility of max-plus algebra has been extensively demonstrated in modeling and optimizing various discrete event systems [2]–[4]. In practical scenarios, this algebraic framework is highly effective for solving complex optimization problems, such as manufacturing systems [5], [6] and railway timetable stability [7]. Furthermore, the evolution of this theory has expanded into broader control systems and model predictive control [8], [9], proving its significance in modern engineering [10], [11]. In recent years, advanced studies have explored the spectral elements and numerical computations of tropical eigenvalues [12], [13], as well as their extremal properties in tropical optimization [14], [15].

The demand for discrete variables in practical applications has triggered the formulation of criteria for integer eigenvectors and subeigenvectors within this structure, as explored in max-linear systems [16], [17]. As its dual, min-plus algebra [18], [19] is used to solve minimization problems [20]–[22]. Despite the advancements in continuous frameworks, finding discrete or integer solutions remains a challenging necessity for real-world resource allocation and scheduling [23], [24].

This article aims to adapt the existence criteria and describe the solution sets of integer eigenvectors and subeigenvectors from max-plus algebra [17] into the min-plus algebra framework. The proposed approach formulates the discrete solutions criteria and proves them mathematically. The innovation of this research lies in establishing the main necessary and sufficient conditions for the existence of integer subeigenvectors and eigenvectors specifically in min-plus algebra, and proposing an algorithmic approach to find the integer image in the column space of the critical matrix.

2. THE COMPREHENSIVE THEORETICAL BASIS

Min-plus algebra $\mathbb{R}_{\min} = (\mathbb{R}_{\varepsilon'}, \oplus', \otimes')$ is a commutative idempotent semifield over the set $\mathbb{R}_{\varepsilon'} = \mathbb{R} \cup \{+\infty\}$, where the minimum operation $(\oplus')$ has the identity $\varepsilon' = +\infty$ and addition $(\otimes')$ has the identity 0 [21].

Min-plus algebra can be extended to matrices with entries in the min-plus algebra. We denote the entry in the i-th row and j-th column of matrix A by $[A]_{ij}$, and its j-th column by $[A]_j$. For matrices, addition is defined for matrices of the same size, say $A, B \in \mathbb{R}_{\varepsilon'}^{m\times n}$, as $[A \oplus' B]_{ij} = \min\{a_{ij}, b_{ij}\}$. Multiplication is defined for matrices with compatible sizes, say $A \in \mathbb{R}_{\varepsilon'}^{m\times k}$ and $B \in \mathbb{R}_{\varepsilon'}^{k\times n}$, as $[A \otimes' B]_{ij} = \min\limits_{\ell=1,2,\ldots,k}\{a_{i\ell} + b_{\ell j}\}$. The identity matrix for multiplication is denoted by I, while the matrix whose entries are all ε' is denoted by ε'. A matrix is said to be row (column) $\mathbb{R}$-astic if each row (column) contains at least one finite element ($< \varepsilon'$). A matrix that is both row and column $\mathbb{R}$-astic is termed doubly $\mathbb{R}$-astic.

Throughout this paper, let $M = \{1,2,\ldots,m\}$ and $N = \{1,2,\ldots,n\}$. Weighted directed graphs $D_A = (N, E, w)$ are typically used to analyze the characteristics of min-plus matrices, where the node set is N, the edge set is $E = \{(i,j) \mid a_{ij} < \varepsilon'\}$, and the edge weights are $w(i,j) = a_{ij}$ [25]. A path $\pi = (v_1, v_2, \ldots, v_{p+1})$ in D_A is a sequence of vertices where each consecutive pair (v_i, v_{i+1}) forms an edge. The length of the path π, denoted by $l(\pi)$, is p, and its weight, denoted by $w(\pi, A)$, is defined as $w(\pi, A) = \sum_{k=1}^{p} a_{v_k v_{k+1}}$. A path with the same starting and ending node is called a circuit. The average weight of a circuit σ is defined as $\mu(\sigma, A) := \frac{w(\sigma,A)}{l(\sigma)}$. The minimum average circuit weight, denotd by $\lambda(A) = \min\limits_{\sigma} \mu(\sigma, A)$, represents the principal eigenvalue of the matrix [18]. A graph is strongly connected if there is a path from any node to any other node. Matrix A is called irreducible if its associated graph (D_A) is strongly connected. Matrix A is called definite if $\lambda(A) = 0$, and strongly definite if it is definite and its main diagonal entries are entirely 0. Algebraically, the minimum weight among all paths of length k from node i to node j is represented by the matrix power $[A^k]_{ij}$ [18].

Theorem 2.1 ([18]). *Let* $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$. *If the average weight of every circuit in* D_A *is not less than zero, then for every integer* $m \geq n$, $A^+ := A \otimes' A^2 \otimes' \cdots = A \otimes' A^2 \otimes' \cdots \otimes' A^m$ *holds and* $A^+ \in \mathbb{R}_{\varepsilon'}^{n\times n}$.

From this result, the matrix A^* is defined as $I \oplus' A^+$, with I being the identity matrix. Furthermore, a vector $x \in \mathbb{R}_{\varepsilon'}^{n}$ $(x \neq \varepsilon')$ is called an eigenvector of A with respect to the eigenvalue λ if $A \otimes' x = \lambda \otimes' x$ and a subeigenvector if $A \otimes' x \geq \lambda \otimes' x$. The search for these involves the scaled matrix $A_\lambda = \lambda(A)^{-1} \otimes' A$ and the critical nodes of the graph:

Definition 2.2. *Given a matrix* $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$, *we denote*

$$N_C(A) = \{i \in N \mid \exists\sigma = (i = i_1, \ldots, i_k, i_1) \in D_A \text{ with } \lambda(A) = \mu(\sigma, A)\}.$$

The elements of $N_C(A)$ *are called the critical nodes of matrix* A.

Theorem 2.3 ([18]). *Let the matrix* $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ *and* $\lambda(A) < \varepsilon'$. *Then* $\lambda(A)$ *is the eigenvalue of* A *and the column* $[A_\lambda^*]_\eta$ *is the eigenvector corresponding to* $\lambda(A)$ *for each* $\eta \in N_C(A)$.

Definition 2.4. *Let* $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$, $\lambda(A) < \varepsilon'$, *and* $N_C(A) = \{i_1, i_2, \ldots, i_k\}$. *The critical columns matrix is defined as*

$$\tilde{A}_\lambda = \begin{bmatrix} | & | & & | \\ [A_\lambda^*]_{i_1} & [A_\lambda^*]_{i_2} & \cdots & [A_\lambda^*]_{i_k} \\ | & | & & | \end{bmatrix}.$$

Furthermore, the algorithmic approach to finding an integer image relies on the foundational theory of min-linear system $A \otimes' x = b$.

Definition 2.5. *Let* $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ *be column* $\mathbb{R}$*-astic and* $b \in \mathbb{R}^m$. *The principal solution of the system* $A \otimes' x = b$ *is denoted by* $\overline{x}(A, b)$ *and is defined by* $[\overline{x}(A,b)]_j := -\min\limits_{i\in M}(a_{ij} - b_i)$ *for each* $j \in N$.

Theorem 2.6 ([19]). *Let* $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ *be column* $\mathbb{R}$*-astic and* $b \in \mathbb{R}^m$. *Then* $A \otimes' x \geq b$ *if and only if* $x \geq \overline{x}(A, b)$.

Theorem 2.7 ([19]). *Let $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ be column $\mathbb{R}$-astic and $b \in \mathbb{R}^m$. Then the system of linear equations $A \otimes' x = b$ has a solution if and only if $\overline{x}(A,b)$ is a solution.*

3. METHOD

This research is a theoretical study utilizing a literature review method and mathematical deductive proofs. The research stages focus on adapting the concepts of discrete eigenvectors and subeigenvectors from max-plus algebra to the min-plus algebra framework, formulating existence criteria and proving theorems, as well as formulating and analyzing the convergence of the algorithm for computing matrix solution spaces.

4. RESULTS AND DISCUSSION

For any matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and scalar $\lambda \in \mathbb{R}$, we denote

$$
\begin{aligned}
&V(A,\lambda) = \{x \in \mathbb{R}^n \mid A \otimes' x = \lambda \otimes' x\}, && V^*(A,\lambda) = \{x \in \mathbb{R}^n \mid A \otimes' x \geq \lambda \otimes' x\},\\
&IV(A,\lambda) = V(A,\lambda) \cap \mathbb{Z}^n, && IV^*(A,\lambda) = V^*(A,\lambda) \cap \mathbb{Z}^n.
\end{aligned}
$$

4.1. INTEGER SUBEIGENVECTORS

The following basic properties are required as a foundation for proving the subsequent propositions.

Lemma 4.1. *If $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ is a row $\mathbb{R}$-astic matrix and $x \in \mathbb{R}^n$, then $A \otimes' x$ is finite.*

Proof. Let $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ be row $\mathbb{R}$-astic and $x \in \mathbb{R}^n$. Since A is row $\mathbb{R}$-astic, for each $i \in M$ there exists $k \in N$ such that $a_{ik} < \varepsilon'$. Since x is finite, $x_k < \varepsilon'$ for each $k \in N$. Thus, for each $i \in M$, $[A \otimes' x]_i = \min\limits_{k\in N}(a_{ik} + x_k) < \varepsilon'$. Therefore, $A \otimes' x$ is finite. ■

Lemma 4.2. *Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ and $k \in \mathbb{N}$.*

(i) It holds that

$$(I \oplus' A)^k = I \oplus' A \oplus' A^2 \oplus' \cdots \oplus' A^k \tag{4.1}$$

(ii) For each $x \in \mathbb{R}_{\varepsilon'}^n$, if $A \otimes' x = x$ then $A^k \otimes' x = x$.

Proof. (i) Clearly, (4.1) is true for $k = 1$. Now assume $(I \oplus' A)^l = I \oplus' A \oplus' A^2 \oplus' \cdots \oplus' A^l$. Then, by the distributive and idempotent properties of matrices, we obtain

$$
\begin{aligned}
(I \oplus' A)^{l+1} &= (I \oplus' A)^l \otimes' (I \oplus' A)\\
&= (I \oplus' A \oplus' A^2 \oplus' \cdots \oplus' A^l) \otimes' (I \oplus' A)\\
&= (I \oplus' A \oplus' A^2 \oplus' \cdots \oplus' A^l) \otimes' I \oplus' (I \oplus' A \oplus' A^2 \oplus' \cdots \oplus' A^l) \otimes' A\\
&= (I \oplus' A \oplus' A^2 \oplus' \cdots \oplus' A^l) \oplus' (A \oplus' A \oplus' A^2 \oplus' \cdots \oplus' A^{l+1})\\
&= I \oplus' A \oplus' A^2 \oplus' \cdots \oplus' A^{l+1}.
\end{aligned}
$$

Thus, it is proven by mathematical induction.

(ii) Assume $A \otimes' x = x$ for any $x \in \mathbb{R}_{\varepsilon'}^n$. It is clear from the assumption that $A^1 \otimes' x = x$. Assume $A^l \otimes' x = x$, then $A^{l+1} \otimes' x = (A^l \otimes' A) \otimes' x = A^l \otimes' (A \otimes' x) = A^l \otimes' x = x$. Therefore, it is proven by mathematical induction. ■

Lemma 4.3. *Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$, $\lambda(A) \geq 0$, and $k \in \mathbb{N}$, then $(A^*)^k = A^*$.*

Proof. By using Theorem 2.1, the distributive property, and the idempotent property, we obtain

$$
\begin{aligned}
A^* \otimes' A^* &= (I \oplus' A^+) \otimes' (I \oplus' A^+)\\
&= (I \oplus' A \oplus' \cdots \oplus' A^n) \otimes' (I \oplus' A \oplus' \cdots \oplus' A^n)
\end{aligned}
$$

$$
\begin{aligned}
&= (I \oplus' A \oplus' \ldots \oplus' A^{2n}) \\
&= (I \oplus' A^{+}) \\
&= A^{*}.
\end{aligned}
$$

For $k = 1$, it is clear that $(A^*)^k = (A^*)^1 = A^*$. Now, assume $(A^*)^k = A^*$ for $k = l$. Note that for $k = l + 1$, it holds that $(A^*)^k = (A^*)^{l+1} = (A^*)^l \otimes' A^* = A^* \otimes' A^* = A^*$. Hence, the lemma is proven by mathematical induction. ■

Once these basic properties are established, we can determine the existence criteria and the form of the solution for finite subeigenvectors.

Proposition 4.4. *Let* $A = (a_{ij}) \in \mathbb{R}_{\varepsilon'}^{n\times n}$ *and* $A \neq \varepsilon'$.

(i) *If* $A \otimes' x \geq \lambda \otimes' x$ *has a finite solution, then* $\lambda \leq \lambda(A)$ *and* $\lambda < \varepsilon'$.

(ii) *If* $\lambda \leq \lambda(A)$ *and* $\lambda < \varepsilon'$, *then* $V^*(A,\lambda) = \{\, (\lambda^{-1} \otimes' A)^* \otimes' u \mid u \in \mathbb{R}^n \,\}$.

(iii) *If* $\lambda \leq \lambda(A)$ *and* $\lambda < \varepsilon'$, *then* $A \otimes' x \geq \lambda \otimes' x \Leftrightarrow x = (\lambda^{-1} \otimes' A)^* \otimes' u, u \in \mathbb{R}^n$.

Proof. (i) Assume $A \otimes' x \geq \lambda \otimes' x, x \in \mathbb{R}^n$ This means that for each $i \in N$, $\min_{j\in N}(a_{ij} + x_j) \geq \lambda + x_i$ holds, or $a_{ij} + x_j \geq \lambda + x_i$ for each $i, j \in N$. Since $A \neq \varepsilon'$, then $\lambda < \varepsilon'$. If $\lambda(A) = \varepsilon'$, then $\lambda < \lambda(A)$. Now suppose $\lambda(A) < \varepsilon'$ then D_A contains a circuit. Let $\sigma = (i_1, \ldots, i_k, i_{k+1} = i_1)$ be any circuit in D_A. Then

$$
\begin{aligned}
a_{i_1 i_2} + x_{i_2} &\geq \lambda + x_{i_1} \\
a_{i_2 i_3} + x_{i_3} &\geq \lambda + x_{i_2} \\
&\vdots \\
a_{i_k i_1} + x_{i_1} &\geq \lambda + x_{i_k}.
\end{aligned}
$$

Summing these inequalities and simplifying gives $\lambda \leq (a_{i_1 i_2} + a_{i_2 i_3} + \ldots + a_{i_{k-1} i_k} + a_{i_k i_1})/k \; = \mu(\sigma, A)$, so $\lambda \leq \min_{\sigma} \mu(\sigma, A) = \lambda(A)$. Conversely, assume $\lambda \leq \lambda(A)$ and $\lambda < \varepsilon'$, then $\lambda(\lambda^{-1} \otimes' A) \leq 0$ and take $u \in \mathbb{R}^n$. It will be shown that $x = (\lambda^{-1} \otimes' A)^* \otimes' u$ satisfies $A \otimes' x \geq \lambda \otimes' x$ and $x \in \mathbb{R}^n$. Since $(\lambda^{-1} \otimes' A)^* \leq I$ then $x \leq u$ and thus $x \in \mathbb{R}^n$. On the other hand, $(\lambda^{-1} \otimes' A)^* \otimes' x = ((\lambda^{-1} \otimes' A)^*)^2 \otimes' u = (\lambda^{-1} \otimes' A)^* \otimes' u = x$ from Lemma 4.3. Therefore, $(\lambda^{-1} \otimes' A) \otimes' x \geq (\lambda^{-1} \otimes' A)^* \otimes' x = x$ so that $A \otimes' x \geq \lambda \otimes' x$.

(ii) Assume $\lambda \leq \lambda(A)$, $\lambda < \varepsilon'$ and $A \otimes' x \geq \lambda \otimes' x$, $x \in \mathbb{R}^n$, then $(\lambda^{-1} \otimes' A) \otimes' x \geq x$ which implies $x \oplus' (\lambda^{-1} \otimes' A) \otimes' x = x$. Thus, $(I \oplus' \lambda^{-1} \otimes' A) \otimes' x = x$, so from Lemma 4.2, $(I \oplus' \lambda^{-1} \otimes' A)^n \otimes' x = x$. From Theorem 2.1 and Equation 4.1, we obtain $(\lambda^{-1} \otimes' A)^* \otimes' x = (I \oplus' \lambda^{-1} \otimes' A)^n \otimes' x = x$. The rest of the proof follows the second part of the proof for (i).

(iii) The proof is the same as in part (ii) except for the reasoning that $x \in \mathbb{R}^n$. ■

Hereafter, we denote $\mathbb{Z}_{\varepsilon'} = \mathbb{Z} \cup \{\varepsilon'\}$. Note that if $\lambda = 0$ and $A \in \mathbb{Z}_{\varepsilon'}^{n\times n}$, then $(\lambda^{-1} \otimes' A)^* \in \mathbb{Z}_{\varepsilon'}^{n\times n}$. Thus, specifically for matrices with integer entries, the criterion can be simplified as follows.

Corollary 4.5. *If* $A \in \mathbb{Z}_{\varepsilon'}^{n\times n}$, *then* $A \otimes' x \geq x$ *has a finite solution* $\Leftrightarrow$ $A \otimes' x \geq x$ *has an integer solution.*

Next, the following theorem provides the main necessary and sufficient condition for the existence of an integer subeigenvector, and describes the form of the solution.

Theorem 4.6. *Let* $A \in \mathbb{R}_{\varepsilon'}^{n\times n}, \lambda \in \mathbb{R}$.

(i) $IV^*(A,\lambda) \neq \emptyset$ *if and only if* $\lambda(\lfloor \lambda^{-1} \otimes' A \rfloor) \geq 0$.

(ii) *If* $IV^*(A,\lambda) \neq \emptyset$, *then* $IV^*(A,\lambda) = \{\, \lfloor \lambda^{-1} \otimes' A \rfloor^* \otimes' z \mid z \in \mathbb{Z}^n \,\}$.

Proof. Assume that $x \in IV^*(A,\lambda)$. Using the fact that $x_i \in \mathbb{Z}$ for each i, we obtain

$$
\begin{aligned}
& A \otimes' x \geq \lambda \otimes' x \\
\Leftrightarrow\;& (\lambda^{-1} \otimes' A) \otimes' x \geq x \\
\Leftrightarrow\;& (\forall i, j \in N) x_i - x_j \leq \lambda^{-1} \otimes' a_{ij}
\end{aligned}
$$

$$\Leftrightarrow\ (\forall i,j \in N) x_i - x_j \leq \lfloor \lambda^{-1} \otimes' a_{ij} \rfloor$$
$$\Leftrightarrow\ \lfloor \lambda^{-1} \otimes' A \rfloor \otimes' x \geq x.$$

Thus, the subeigenvector of A corresponding to λ is exactly the integer subeigenvector of $\lfloor \lambda^{-1} \otimes' A \rfloor \in \mathbb{Z}_{\varepsilon'}^{n\times n}$ corresponding to 0.

(i) Now, from Proposition 4.4, the integer subeigenvector of $\lfloor \lambda^{-1} \otimes' A \rfloor \in \mathbb{Z}_{\varepsilon'}^{n\times n}$ corresponding to $\lambda = 0$ exists if and only if $\lambda(\lfloor \lambda^{-1} \otimes' A \rfloor) \leq 0$. Furthermore, $\lfloor \lambda^{-1} \otimes' A \rfloor \in \mathbb{Z}_{\varepsilon'}^{n\times n}$ so according to Corollary 4.5, it is concluded that a finite subeigenvector exists if and only if an integer subeigenvector exists.

(ii) If a finite subeigenvector exists, then from Proposition 4.4 we obtain $V^*(\lfloor \lambda^{-1} \otimes' A \rfloor, 0) = \{\lfloor \lambda^{-1} \otimes' A \rfloor^* \otimes' u \mid u \in \mathbb{R}^n\}$. However, since $\lfloor \lambda^{-1} \otimes' A \rfloor$ is an integer matrix, $\lfloor \lambda^{-1} \otimes' A \rfloor^*$ is also an integer matrix, meaning that all integer subeigenvectors can be described by taking the min-combination of the columns of $\lfloor \lambda^{-1} \otimes' A \rfloor^*$ with integer coefficients. Note that integer vectors might be formed from the min-combination of integer columns of a matrix with real coefficients, but only if the real coefficients correspond to inactive columns. However, any integer vector formed in this way can be formed using integer coefficients, for example, by applying the ceiling function to its coefficients. Thus, it is sufficient to take only integer coefficients. ■

4.2. INTEGER EIGENVECTORS

Turning to the eigenvector problem, the following proposition asserts the uniqueness of the eigenvalue for a finite eigenvector.

Proposition 4.7. *Let* $A \in \mathbb{R}_{\varepsilon'}^{n\times n}, \lambda \in \mathbb{R}_{\varepsilon'}$, *and* $x \in \mathbb{R}^n$. *If* $A \otimes' x = \lambda \otimes' x$, *then* $\lambda = \lambda(A)$.

Proof. Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}, x \in \mathbb{R}^n, \lambda \in \mathbb{R}$, and $A \otimes' x = \lambda \otimes' x$. If $A = \varepsilon'$, then the left side of $A \otimes' x = \lambda \otimes' x$ is ε' so the only satisfying value for λ is ε' and clearly $\lambda(A) = \varepsilon'$. Now assume $A \neq \varepsilon'$. Then the left side of $A \otimes' x = \lambda \otimes' x$ is not ε', so $\lambda < \varepsilon'$, which means D_A contains a circuit. The equation $A \otimes' x = \lambda \otimes' x$ means that for each $i \in N$ it holds that $\min_{j\in N}(a_{ij} + x_j) = \lambda + x_i$ or $a_{ij} + x_j \geq \lambda + x_i$ or $a_{ij} - \lambda \geq x_i - x_j$ for each $i,j \in N$. Let $\sigma = (i_1, i_2, \dots, i_{p+1} = i_1)$ be any circuit in D_A. Summing the inequalities along σ gives

$$\sum_{k=1}^{p}(a_{i_k i_{k+1}} - \lambda) \geq \sum_{k=1}^{p}(x_{i_k} - x_{i_{k+1}}) = 0 \Rightarrow w(\sigma, A) \geq p \cdot \lambda \Rightarrow \mu(\sigma, A) = \frac{w(\sigma, A)}{p} \geq \lambda.$$

Since σ is any circuit in D_A, then $\lambda \leq \min_{\sigma} \mu(\sigma, A) = \lambda(A)$. Furthermore, the equation $\min_{j\in N}(a_{ij} + x_j) = \lambda + x_i$ means that for each $i \in N$ there is an index $p(i)$ satisfying $a_{i\,p(i)} + x_{p(i)} = \lambda + x_i$ or

$$a_{i\,p(i)} - \lambda = x_i - x_{p(i)} \tag{4.2}$$

Construct a sequence of indices $(i_1, i_2, \dots)$ with $i_{k+1} = p(i_k)$ for each k. Since n is finite, the sequence must contain a circuit, say circuit $\pi = (i_p, i_{p+1}, \dots, i_{p+q} = i_p)$. By summing equations (4.2) along π, we get $\sum_{k=p}^{p+q-1}(a_{i_k i_{k+1}} - \lambda) = 0 \Rightarrow \lambda = \frac{w(\pi,A)}{q} = \frac{w(\pi,A)}{l(\pi)} = \mu(\pi, A)$. Since $\lambda(A)$ is the minimum average weight of all circuits, then $\lambda \geq \lambda(A)$. Therefore, it must be that $\lambda = \lambda(A)$. ■

The following algebraic property is needed to construct finite eigenvectors from the matrix A_λ^*.

Lemma 4.8. *Let* $A \in \mathbb{R}_{\varepsilon'}^{m\times n}, \lambda(A) < \varepsilon'$. *If* $x \in \mathbb{R}^n$ *is an eigenvector of* A, *then* $A_\lambda^* \otimes' x = x$

Proof. Assume $A \in \mathbb{R}_{\varepsilon'}^{n\times n}, \lambda(A) < \varepsilon', x \in \mathbb{R}^n$ and $A \otimes' x = \lambda \otimes' x$. From Proposition 4.7, $\lambda = \lambda(A)$ Thus $A \otimes' x = \lambda(A) \otimes' x$, so $A_\lambda \otimes' x = (\lambda(A)^{-1} \otimes' A) \otimes' x = x$. From Lemma 4.2, $(A_\lambda)^k \otimes' x = x$ for each $k \in \mathbb{N}$, and using Theorem 2.1 we obtain

$$\begin{aligned} A_\lambda^* \otimes' x &= (I \oplus' A \oplus' \dots \oplus' A^n) \otimes' x \\ &= (I \otimes' x) \oplus' (A \otimes' x) \oplus' \dots \oplus' (A^n \otimes' x) \end{aligned}$$

$$= x \oplus' x \oplus' \cdots \oplus' x$$
$$= x$$

by the idempotent property. ■

Based on the lemma above, finite eigenvectors can be expressed as combinations of the columns of the critical matrix.

Proposition 4.9. *Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ with $\lambda(A) < \varepsilon'$. If $x \in \mathbb{R}^n$ is an eigenvector of A, then*

$$x = \oplus'_{c\in N_C(A)} \left(x_c \otimes' [A_\lambda^*]_c\right).$$

Proof. Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}, \lambda(A) < \varepsilon', x \in \mathbb{R}^n$ and x is an eigenvector of A. Let $y = \oplus'_{c\in N_C(A)} (x_c \otimes' [A_\lambda^*]_c)$. Then for each $i \in N$, $y_i = \min\limits_{c\in N_C(A)}([A_\lambda^*]_{ic} + x_c)$. Since x is an eigenvector, according to Lemma 4.8, then $x = A_\lambda^* \otimes' x$ or $x_i = \min\limits_{j\in N}\left([A_\lambda^*]_{ij} + x_j\right)$. Note that $N_C(A) \subseteq N$, so $y_i = \min\limits_{c\in N_C(A)}([A_\lambda^*]_{ic} + x_c) \geq \min\limits_{j\in N}\left([A_\lambda^*]_{ij} + x_j\right) = x_i$, giving $y \geq x$. Now, take any index $i_0 \in N$. Since x is an eigenvector, there exists an index $i_1 \in N$ such that $x_{i_0} = [A_\lambda]_{i_0 i_1} + x_{i_1}$. By repeating this argument, we can construct a sequence of indices $(i_0, i_1, i_2, \ldots)$ satisfying

$$x_{i_k} = [A_\lambda]_{i_k i_{k+1}} + x_{i_{k+1}} \tag{4.3}$$

Since n is finite, the sequence contains a circuit, say circuit $\gamma = \left(i_p, i_{p+1}, \ldots, i_q = i_p\right)$. Summing equations (4.3) along the circuit γ gives $x_{i_p} - x_{i_q} = w(\gamma, A_\lambda) \Rightarrow 0 = w(\gamma, A_\lambda)$. Since the circuit weight is 0, every node in γ is a critical node, $i_p, i_{p+1}, \ldots, i_{q-1} \in N_C(A)$. Choose a node $c \in \left\{i_p, i_{p+1}, \ldots, i_{q-1}\right\}$. Then by constructing a path ρ from i_0 to c and summing equations (4.3) along ρ, we get $x_{i_0} = w(\rho, A_\lambda) + x_c$. By definition, $[A_\lambda^*]_{i_0 c} \leq w(\rho, A_\lambda)$ so $x_{i_0} \geq [A_\lambda^*]_{i_0 c} + x_c$. Since $c \in N_C(A)$, the term on the right side is one of the minimized elements in the definition of y_{i_0} so $x_{i_0} \geq \min\limits_{k\in N_C(A)}\left([A_\lambda^*]_{i_0 k} + x_k\right) = y_{i_0}$. Since i_0 is any member in N, then $x \geq y$. Therefore, it must be that $x = y$. ■

From this decomposition, we obtain an existence criterion for finite eigenvectors that depends on the row property of the matrix $\tilde{A}_\lambda$.

Proposition 4.10. *Let $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ and $\lambda(A) < \varepsilon'$. Then $V(A, \lambda) \neq \emptyset$ if and only if $\lambda = \lambda(A)$ and $\tilde{A}_\lambda$ is a row $\mathbb{R}$-astic matrix. If $V\left(A, \lambda(A)\right) \neq \emptyset$, then $V\left(A, \lambda(A)\right) = \left\{\tilde{A}_\lambda \otimes' u \mid u \in \mathbb{R}^k\right\}$ with A_λ of size $n \times k$.*

Proof. Assume $V(A, \lambda) \neq \emptyset$. Then there exists $x \in \mathbb{R}^n$ which is an eigenvector of A. From Proposition 4.7, then $\lambda = \lambda(A)$. Next, from Proposition 4.9, $x = \oplus'_{c\in N_C(A)} (x_c \otimes' [A_\lambda^*]_c)$. Since x is finite, for each $i \in N$, x_i must be finite so that $\min\limits_{c\in N_C(A)}(x_c \otimes' [A_\lambda^*]_{ic}) < \varepsilon'$. Thus, for each $i \in N$ there must be $c \in N_C(A)$ such that $[A_\lambda^*]_{ic} < \varepsilon'$. According to Definition 2.4, for each $c \in N_C(A)$, $[A_\lambda^*]_c$ is a column of $\tilde{A}_\lambda$. Therefore, $\tilde{A}_\lambda$ is row $\mathbb{R}$-astic. Conversely, assume $\lambda = \lambda(A)$ and $\tilde{A}_\lambda$ is a row $\mathbb{R}$-astic matrix. From Theorem 2.3, for each $j \in |N_C(A)|$, $\left[\tilde{A}_\lambda\right]_j$ is an eigenvector such that $A \otimes' \tilde{A}_\lambda = \lambda(A) \otimes' A_\lambda$. Thus, there exists $u = (0,0,\ldots,0)^T \in \mathbb{R}^{|N_C(A)|}$ such that $v = \tilde{A}_\lambda \otimes' u$ is a finite vector (according to Lemma 4.1) and satisfies

$$A \otimes' v = A \otimes' \left(\tilde{A}_\lambda \otimes' u\right) = \left(A \otimes' \tilde{A}_\lambda\right) \otimes' u = \tilde{A}_\lambda \otimes' u = v.$$

The second part of the proof can be shown using the same argument as the previous part. ■

The following results are needed to analyze the critical node structure and guarantee boundedness of values in the related matrices.

Proposition 4.11. *Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$. Then $\eta \in N_C(A) \Leftrightarrow [A_\lambda^+]_{\eta\eta} = 0$.*

Proof. Recall that $A_\lambda = \lambda(A)^{-1} \otimes' A$. Then D_{A_λ} has the same edge structure as D_A but the weight of each edge (i,j) is reduced by $\lambda(A)$. The weight of circuit σ in graph D_{A_λ} is $w(\sigma, A_\lambda) = w(\sigma, A) - l(\sigma) \cdot \lambda(A)$. Since $\lambda(A)$ is the minimum average circuit weight, for each circuit σ it holds that $\frac{w(\sigma,A)}{l(\sigma)} \geq \lambda(A)$, which implies

$$w(\sigma, A) - l(\sigma) \cdot \lambda(A) \geq 0\,.$$

Thus, there are no negative weight circuits in D_{A_λ}. Assume $\eta \in N_C(A)$. From Definition 2.2, η is a critical node, meaning η is traversed by some critical circuit, say σ, which has an average weight equal to $\lambda(A)$. Consequently, $\mu(\sigma, A) = \frac{w(\sigma,A)}{l(\sigma)} = \lambda(A)$, so $w(\sigma, A) - l(\sigma) \cdot \lambda(A) = 0$ meaning the weight of circuit $\sigma \in D_{A_\lambda}$ is 0. The entry $[A_\lambda^+]_{\eta\eta}$ represents the minimum path weight from η to η with length $k \geq 1$. Since a circuit σ passing through η with weight 0 has been found, and it is known there are no negative weight circuits in D_{A_λ}, its minimum weight must be exactly 0. So, $[A_\lambda^+]_{\eta\eta} = 0$.

Now, assume $[A_\lambda^+]_{\eta\eta} = 0$. This means there is some circuit σ from η to η in D_{A_λ} with total weight 0. Thus, $w(\sigma, A) - l(\sigma) \cdot \lambda(A) = w(\sigma, A_\lambda) = 0$, so $\frac{w(\sigma,A)}{l(\sigma)} = \lambda(A)$ which means σ has an average weight equal to $\lambda(A)$. In other words, σ is a critical circuit. Hence, $\eta \in N_C(A)$. ■

Thus, the matrix $\tilde{A}_\lambda$ can be constructed from the columns of A_λ^+ whose diagonal entries are 0. In addition, $\tilde{A}_\lambda$ can also be formed from identical columns in A_λ^+ and A_λ^*.

Lemma 4.12. *Let* $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ *and* $\lambda(A) < \varepsilon'$.
(i) $\tilde{A}_\lambda$ *is a column* $\mathbb{R}$*-astic matrix.*
(ii) If A *is irreducible, then* A_λ^+ *and* $\tilde{A}_\lambda$ *are finite.*

Proof. (i) From Definition 2.4 and Proposition 4.11, it is clear that there are 0 entries in every column of $\tilde{A}_\lambda$.
(ii) Let matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ be irreducible, which means D_A is a strongly connected graph. Matrix $A_\lambda = \lambda(A)^{-1} \otimes' A$. Because $\lambda(A)$ is finite, the scaling operation does not change the position of the ε' entries. That is, $[A_\lambda]_{ij} < \varepsilon'$ if and only if $a_{ij} < \varepsilon'$. Therefore, the graph structure of D_{A_λ} is identical to D_A. Consequently, D_{A_λ} is also strongly connected. Since D_{A_λ} is strongly connected, for every pair of nodes (i,j) there must be a path from i to j. In other words, for each $i, j \in N$ there exists $k \in N$ such that $\left[A_\lambda^k\right]_{ij} < \varepsilon'$. From Theorem 2.1,

$$[A_\lambda^+]_{ij} = [A_\lambda]_{ij} \oplus' [A_\lambda^2]_{ij} \oplus' \cdots \oplus' [A_\lambda^n]_{ij}.$$

Thus $[A_\lambda^+]_{ij} < \varepsilon'$ for each $i, j \in N$, meaning the matrix A_λ^+ is finite. Furthermore, it is clear that the matrix $\tilde{A}_\lambda$ is also finite because it is constructed from columns of the A_λ^+. ■

In the special case when the matrix is strongly definite, the solution structure becomes simpler because its eigenvalue is zero.

Proposition 4.13. *If* $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ *is a strongly definite matrix, then* $A^+ = A^* = \tilde{A}_\lambda$.

Proof. Since $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ is strongly definite, $\lambda(A) = 0$ and $a_{ii} = 0$ for each $i \in N$. Consequently, $A_\lambda = A$. Take any $i \in N$. Note that $[A^k]_{ii}$ represents the minimum weight of paths with length k from i to i. One of the paths with length k from i to i is $(i, i, \ldots, i)$ which has weight $k \times 0 = 0$, so $[A^k]_{ii} \leq 0$. On the other hand, since $\lambda(A) = 0$ the average weight of any circuit in $D_A \geq 0$, meaning the total weight of any circuit ≥ 0. That is, $[A^k]_{ii} \geq 0$. Thus $[A^k]_{ii} = 0$ so $[A^+]_{ii} = [A \oplus' A^2 \oplus' \cdots \oplus' A^n]_{ii} = [A]_{ii} \oplus' [A^2]_{ii} \oplus' \ldots \oplus' [A^n]_{ii} = 0$. Therefore, $A^+ = \tilde{A}_\lambda$ and

$$[A^*]_{ij} = [I \oplus' A^+]_{ij} = \begin{cases} 0 \oplus' [A^+]_{ii} = 0 \oplus' 0 = 0 & \text{if } i = j \\ \varepsilon' \oplus' [A^+]_{ij} = [A^+]_{ij} & \text{if } i \neq j \end{cases} = [A^+]_{ij}$$ ■

Corollary 4.14. *If* A *is a strongly definite matrix, then* $V(A,0) = V^*(A,0)$.

Proof. Proved directly from Propositions 4.4 and 4.10. ■

Based on Proposition 4.7, $IV(A,\lambda) \neq \emptyset$ only if $\lambda = \lambda(A)$. Thus, without loss of generality, $IV\big(A,\lambda(A)\big)$ is denoted as $IV(A)$. From Theorem 4.6 and Corollary 4.14, the solution to the integer eigenvector problem for strongly definite matrices can be obtained.

Corollary 4.15. *Let* $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ *be a strongly definite matrix.*
(i) $IV(A) \neq \emptyset$ *if and only if* $\lambda(\lfloor A \rceil) = 0$.
(ii) *If* $IV(A) \neq \emptyset$ *then* $IV(A) = \{\, \lfloor A \rceil^* \otimes' z \mid z \in \mathbb{Z}^n \,\}$.

Proof. Let $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ be a strongly definite matrix. From Corollary 4.14, it is known that $V(A,0) = V^*(A,0)$, so $IV(A) = IV(A,0) = V(A,0) \cap \mathbb{Z}^n = V^*(A,0) \cap \mathbb{Z}^n = IV^*(A,0)$. (i) Assume $IV(A) = IV^*(A,0) \neq \emptyset$. From Theorem 4.6, it is obtained that $\lambda(\lfloor A \rceil) \geq 0$. On the other hand, it is clear that $A \geq \lfloor A \rceil$. Since $\lambda(\cdot)$ is monotonic, $\lambda(A) \geq \lambda(\lfloor A \rceil)$. Then, since A is strongly definite, $\lambda(A) = 0$ meaning $0 \geq \lambda(\lfloor A \rceil)$. Thus, $\lambda(\lfloor A \rceil) = 0$. Conversely, directly from Theorem 4.6, $\lambda(\lfloor A \rceil) = 0 \Rightarrow \lambda(\lfloor A \rceil) \leq 0 \Rightarrow IV^*(A,0) \neq \emptyset$.

(ii) From Theorem 4.6, $IV(A) = IV^*(A,0) \neq \emptyset \Rightarrow IV(A) = IV^*(A,0) = \{\lfloor A \rceil^* \otimes' z | z \in \mathbb{Z}^n\}$. ■

Next, the special case for integer matrices (matrices over $\mathbb{Z}_{\varepsilon'}$) is examined.

Proposition 4.16. *Let* $A \in \mathbb{Z}_{\varepsilon'}^{n\times n}$. *Then* A *has an integer eigenvector if and only if* $\lambda(A) \in \mathbb{Z}$ *and* $\tilde{A}_\lambda$ *is a row* $\mathbb{R}$-*astic matrix.*

Proof. First assume that $x \in IV(A)$. From Proposition 4.7, it is known that the only eigenvalue corresponding to x is $\lambda(A)$. Then $A \otimes' x = \lambda(A) \otimes' x$, with the product on the left side being an integer. To ensure the right side is also an integer, it must be that $\lambda(A) \in \mathbb{Z}$. Furthermore, every integer eigenvector is a finite vector, meaning $\tilde{A}_\lambda$ is a row $\mathbb{R}$-astic matrix from Proposition 4.10. Now assume that $\lambda(A) \in \mathbb{Z}$ and A_λ is a row $\mathbb{R}$-astic matrix. Then $A_\lambda \in \mathbb{Z}_{\varepsilon'}^{n\times n}$ meaning every entry of A_λ^+, A_λ^*, and $\tilde{A}_\lambda$ is a member of $\mathbb{Z}_{\varepsilon'}$. From Proposition 4.9, it is also known that every finite eigenvector can be described as a min-combination of columns of $\tilde{A}_\lambda$. According to Lemma 4.1, we can pick integer coefficients to obtain an integer eigenvector of A. ■

Corollary 4.17. *An irreducible matrix* $A \in \mathbb{Z}_{\varepsilon'}^{n\times n}$ *has an integer eigenvector if and only if* $\lambda(A) \in \mathbb{Z}$.

Proof. Assume the matrix $A \in \mathbb{Z}_{\varepsilon'}^{n\times n}$ is irreducible. Since A is irreducible, according to Lemma 4.12, $\tilde{A}_\lambda$ is finite, meaning A_λ is a row $\mathbb{R}$-astic matrix. Subsequently, it is proven directly from Proposition 4.16. ■

However, it is worth noting that the results above do not generally apply to any arbitrary matrix $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$. Nevertheless, there is a necessary condition regarding matrix elements for integer solutions to exist.

Proposition 4.18. *Let* $A \in \mathbb{R}_{\varepsilon'}^{n\times n}$ *be a matrix such that it has an integer eigenvector corresponding to an integer eigenvalue, then* A *has at least one integer entry in each of its rows.*

Proof. It is already known that the only eigenvalue corresponding to an integer eigenvector is $\lambda(A)$, and thus from the assumption, $\lambda(A) \in \mathbb{Z}$. Now suppose $x \in IV(A)$. Then $A \otimes' x = \lambda(A) \otimes' x$, with the right side being an integer so that $(\forall i \in N)\min\big(a_{ij} + x_j\big) \in \mathbb{Z}$, which implies that for each $i \in N$ there exists an index j such that $a_{ij} \in \mathbb{Z}$. ■

Unfortunately, there appears to be no simple answer to determining the existence of integer eigenvectors in general. However, Proposition 4.10 shows that this problem can be solved by looking for integer vectors in a finitely generated subspace (namely the column space of matrix $\tilde{A}_\lambda$).

Now, the question of whether there exists an integer vector z in the column space of A, which will be referred to as an integer image of A for a matrix $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ will be discussed. Hereafter, denoted as $IIm(A) = \{z \in \mathbb{Z}^m \mid (\exists x \in \mathbb{R}_{\varepsilon'}^{n}) A \otimes' x = z\}$. Note that if $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ has an ε' row, then $IIm(A) = \emptyset$, and if A has an ε' column, then $IIm(A) = IIm(A')$ where A' is obtained from matrix A by removing the ε' column. Therefore, in this section it suffices to review matrices that are doubly ℝ-astic. To solve this problem, the following algorithm is proposed:

INT-IMAGE Algorithm

```
Input: Doubly ℝ-astic matrix $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ and any initial vector $x^{(0)} \in \mathbb{Z}^m$.
Output: Vector $x \in IIm(A)$ or an indication that no integer image exists.
(1) $r := 1$.
(2) $z := \bar{x}(A, x^{(r-1)}),\ y := A \otimes' z$.
(3) If $y \in \mathbb{Z}^m$ STOP: $y \in IIm(A)$ .
(4) $x_i^{(r)} := \lceil y_i \rceil$ for each $i \in M$.
(5) If $x_i^{(r)} > x_i^{(0)}$ for each $i \in M$ STOP: No integer image exists.
(6) $r := r + 1$. Go to (2).
```

Note that every vector generated by the INT-IMAGE Algorithm is a finite vector because from Lemma 4.1 and the fact that $\bar{x}(A, u)$ is finite if u is finite since A is a doubly ℝ-astic matrix.

Lemma 4.19. *Let* $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ *and* $x, y \in \mathbb{R}_{\varepsilon'}^{n}$, *then* $x \geq y \Rightarrow A \otimes' x \geq A \otimes' y$.

Proof. Assume $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ and $x, y \in \mathbb{R}_{\varepsilon'}^{n}$. Note that for each $j \in N$ and each $i \in M$, it holds that

$$\begin{aligned} & x_j \geq y_j \\ \Rightarrow\ & a_{ij} + x_j \geq a_{ij} + y_j \\ \Rightarrow\ & \min_{j\in N}(a_{ij} + x_j) \geq \min_{j\in N}(a_{ij} + y_j) \\ \Rightarrow\ & [A \otimes' x]_i \geq [A \otimes' y]_i. \end{aligned}$$

Thus, $A \otimes' x \geq A \otimes' y$. ■

Lemma 4.20. *Let* $u \in \mathbb{R}^m$ *be an image of a column ℝ-astic matrix* $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$, *then* $A \otimes' \bar{x}(A, u) = u$.

Proof. Since $u \in \mathbb{R}^n$ is an image of $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ then $A \otimes' v = u$ for some v. According to Proposition 2.7, $\bar{x}(A, u)$ is a solution to $A \otimes' v = u$ or $A \otimes' \bar{x}(A, u) = u$. ■

Lemma 4.21. *Let* $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ *be column ℝ-astic and* $x, y \in \mathbb{R}^m$, *then* $x \leq y \Rightarrow A \otimes' \bar{x}(A, x) \leq A \otimes' \bar{x}(A, y)$.

Proof. Assume $A \in \mathbb{R}_{\varepsilon'}^{m\times n}$ be column ℝ-astic, $x, y \in \mathbb{R}^m$, and $x \leq y$. Then for each $i \in M$ and each $j \in N$,

$$\begin{aligned} & x_i \leq y_i \\ \Rightarrow\ & -x_i \geq -y_i \\ \Rightarrow\ & a_{ij} - x_i \geq a_{ij} - y_i \\ \Rightarrow\ & \min_{i\in M}(a_{ij} - x_i) \geq \min_{i\in M}(a_{ij} - y_i) \\ \Rightarrow\ & -\min_{i\in M}(a_{ij} - x_i) \leq -\min_{i\in M}(a_{ij} - y_i) \end{aligned}$$

so $[\bar{x}(A, x)]_j \leq [\bar{x}(A, y)]_j$ from Definition 2.5. Thus, $\bar{x}(A, x) \leq \bar{x}(A, y)$ so that $A \otimes' \bar{x}(A, x) \leq A \otimes' \bar{x}(A, y)$ according to Lemma 4.19. ■

Based on these properties, the following is a proof that the sequence generated by the INT-IMAGE Algorithm is non-decreasing and bounded.

Claim 4.22. *The sequence* $\{x^{(r)}\}_{r=0,1,\ldots}$ *is a non-decreasing sequence.*

Proof. Note that for each $x^{(r)}$ the algorithm attempts to solve the system of equations $A \otimes' x = x^{(r)}$ by determining $z = \overline{x}(A, x^{(r)})$ which, according to Proposition 2.6, satisfies $A \otimes' z \geq x^{(r)}$. If equality occurs, the algorithm stops (since $x^{(r)}$ is an integer vector), if not, the algorithm calculates

$$x^{(r+1)} = \lceil A \otimes' z \rceil \geq A \otimes' z \geq x^{(r)}$$ ■

Claim 4.23. *If A has an integer image, then the sequence* $\{x^{(r)}\}_{r=0,1,\ldots}$ *is bounded above by a vector in* $IIm(A)$.

Proof. Assume $u \in IIm(A)$. Then $\gamma \otimes' u \in IIm(A)$ for each $\gamma \in \mathbb{Z}$. Choose γ large enough so $\gamma \otimes' u \geq x^{(0)}$. Now assume that $x^{(r)} \leq v$ for some $v \in IIm(A)$. Then from Lemmas 4.20 and 4.21, we get

$$x^{(r+1)} = \left\lceil A \otimes' \overline{x}(A, x^{(r)}) \right\rceil \leq \lceil A \otimes' \overline{x}(A, v) \rceil = \lceil v \rceil = v.$$

Hence, the claim is proven by mathematical induction. ■

Claim 4.24. *If* $x_i^{(r)} > x_i^{(0)}$ *for some* r *and every* i, *then* A *does not have an integer image.*

Proof. Suppose $u \in IIm(A)$. According to Claims 4.22 and 4.23, the sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ is non-decreasing and bounded above. However, from the proof of Claim 4.23, it can be observed that we can choose $\gamma \in \mathbb{Z}$ such that: (i) $\gamma \otimes' u \in IIm(A)$, (ii) $\gamma \otimes' u \geq x^{(r)}$ for each r, and (iii) there exists an i such that $(\gamma \otimes' u)_i = x_i^{(0)}$. Thus, we get $x_i^{(0)} = (\gamma \otimes' u)_i \geq x_i^{(r)} \geq x_i^{(0)}$, which implies that the i-th entry of every $x^{(r)}$ is always the same, and therefore there is no iteration where all entries of $x^{(r)}$ strictly increase. ■

Finally, the convergence of this algorithm is guaranteed by the following theorem, which also serves as the final criterion for the existence of an integer image.

Theorem 4.25. *The doubly* $\mathbb{R}$*-astic input matrix* $A \in \mathbb{R}_{\varepsilon'}^{m \times n}$ *has an integer image if and only if the sequence* $\{x^{(r)}\}_{r=0,1,\ldots}$ *generated by the INT-IMAGE Algorithm is finitely convergent.*

Proof. If the matrix has an integer image, then from Claims 4.22 and 4.23, the sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ is non-decreasing and bounded above by some integer image x of A. Consequently, the sequence $\{x^{(r)}\}_{r=0,1,\ldots}$ will converge. Furthermore, since the sequence is a sequence of integer vectors, at every step at least one entry must strictly increase (increase by at least one) until it finally reaches the value of the corresponding x. Therefore, its convergence must be finite.

If the sequence is finitely convergent, then there is an s such that for each $r \geq s$, $x^{(r)} = x^{(r+1)}$. Consequently, $y = A \otimes' \overline{x}(A, x^{(s)}) \in \mathbb{Z}^m$. Because if we suppose $y \notin \mathbb{Z}^m$, then there is an entry i of y that is not an integer, meaning $y_i > x_i^{(s)}$. But if so, then $x_i^{(s+1)} = \lceil y_i \rceil > y_i > x_i^{(s)}$ which is a contradiction. So, it must be that $y \in IIm(A)$. ■

5. CONCLUSION

This research concludes that for a matrix $A \in \mathbb{R}_{\varepsilon'}^{n \times n}$ with $A \neq \varepsilon'$, an integer subeigenvector corresponding to a scalar λ exists if and only if it satisfies the condition $\lambda(\lfloor \lambda^{-1} \otimes' A \rfloor) \geq 0$. If this existence criterion is satisfied, the solution set can be formulated as $IV^*(A, \lambda) = \{\, \lfloor \lambda^{-1} \otimes' A \rfloor^* \otimes' z \mid z \in \mathbb{Z}^n \,\}$. Meanwhile, an integer eigenvector can only correspond to its principal eigenvalue, namely $\lambda(A)$. For a strongly definite matrix A, an integer eigenvector exists if and only if $\lambda(\lfloor A \rfloor) = 0$ with the solution form $IV(A) =$

$\{\lfloor A\rfloor^* \otimes' z \mid z \in \mathbb{Z}^n\}$. On the other hand, for an integer matrix A, an integer eigenvector exists if and only if $\lambda(A) \in \mathbb{Z}$ and the matrix $\tilde{A}_\lambda$ is row $\mathbb{R}$-astic. In general, determining an integer eigenvector is equivalent to finding an integer image in the column space of the matrix $\tilde{A}_\lambda$, which can be solved using the INT-IMAGE algorithm.

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