Dynamics of Solow-Spatial Model with Capital-Augmenting Technology and Human Capital

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Abstract: *This paper investigates the dynamics of a spatial Solow growth model incorporating capital-augmenting technology and human capital. We establish that the economically relevant steady state remains locally asymptotically stable under spatial diffusion. Furthermore, our stability and bifurcation analysis shows that the system does not exhibit Turing or Hopf bifurcations, as the determinant of the spatial Jacobian remains strictly positive for all wave numbers ($p \geq 0$). Consequently, spatial diffusion consistently acts to smooth out regional disparities over time, driving the system towards a uniform steady state without being constrained by a specific diffusion-ratio threshold.*
Keywords: *Solow model, capital augmenting-technology, human capital, reaction-diffusion, steady state*

1. INTRODUCTION

In modern economics, economic growth is determined not only by the accumulation of physical capital, but also by the quality of human resources and technological advances that affect production efficiency [1]. The neoclassical growth model by Solow [2] and Swan [3] explains that the interaction between capital, labor, and technology determines long-term national income dynamics [4]. These factors of production are represented by a production function. One example is the Cobb–Douglas production function, which illustrates the contribution of each factor to output [5].

Technological progress is generally assumed to be neutral with respect to factors of production [6]. However, empirical findings by Boskin and Lau [7] show that technological progress is capital–augmenting, i.e., it increases the productivity of physical capital, making this approach relevant in modeling modern economic growth. Alongside physical capital and technology, the role of human capital is equally pivotal. Countries that focus on human resource development generally show higher increases in per capita income, Bluedorn [8]. The quality of the workforce, as measured by skills and knowledge, allows for an understanding of variations in output across countries. Furthermore, human capital is not just an input. It is fundamental to a region's ability to adopt technological innovations [9] and is a primary engine for sustained endogenous growth [10]. So, consistent with that approach and the augmented Solow model by Mankiw et al. [11], the model used in this study assumes that the production process follows the Cobb–Douglas production function involving human capital, which is expressed as

$$Y(t) = \big(A(t)K(t)\big)^{\alpha} H(t)^{\beta} L(t)^{1-\alpha-\beta}, \tag{1}$$

where $\alpha, \beta \in (0,1), \alpha + \beta < 1$, $K(t)$ denotes physical capital, $H(t)$ denotes human capital, $L(t)$ denotes the labor force, and $A(t)$ denotes the level of technology. The parameters α and β represent the elasticity of output with respect to physical and human capital, respectively. Technology is assumed to increase the productivity of physical capital. This means that changes in technology have a direct effect on the efficiency of capital use in the production process. The dynamics of physical capital and human capital accumulation are given by

$$\begin{aligned} \dot{K}(t) &= s_K\, Y(t) - \delta\, K(t), \\ \dot{H}(t) &= s_H\, Y(t) - \delta\, H(t), \end{aligned} \tag{2}$$

where s_K and s_H represent the proportions of output reinvested in physical capital and human capital, respectively, while δ represents the rate of capital depreciation. Labor and technology are assumed to grow exponentially at constant rates of n and g, respectively.

Classical growth models typically assume homogeneous economic conditions across space. However, the distribution of physical capital and human resources between regions is uneven in reality, which gives rise to spatial dynamics that can affect the stability of economic systems [12, 13, 14]. Differences in capital accumulation levels between regions can trigger capital migration, which can be mathematically represented through a diffusion process [15, 16]. Such continuous spatial dynamics have been formulated to rigorously capture the flow of capital and labor across geographical boundaries [17]. Furthermore, analyzing the local and global stability of these spatial economic models shows how diffusion significantly affects regional convergence over time [18, 19]. According to research by Sutrima et al. [20], diffusion can impact the existence and stability of the system's steady states. Similar results were demonstrated by Sutrima and Setyowati [21] in a diffusive wage-employment system, where the existence of global solutions, positivity, and equilibrium stability are significantly influenced by diffusion parameters and nonlinear interactions.

Based on these considerations, the economic growth model is augmented with a spatial framework under the assumption that physical and human capital depend on space and time. By incorporating diffusion effects, model (1) is represented in a system of partial differential equations as

$$\begin{aligned} k_t &= D_k\, k_{xx}(x,t) + f_1(k,h), (x,t) \in (0,\ell) \times (0,\infty), \\ h_t &= D_h\, h_{xx}(x,t) + f_2(k,h), (x,t) \in (0,\ell) \times (0,\infty), \end{aligned} \tag{3}$$

where

$$f_1(k,h) = s_K\, k^\alpha h^\beta - (\delta + n + ag)k, \qquad f_2(k,h) = s_H k^\alpha h^\beta - (\delta + n + bg)h,$$
$$a = \frac{1-\beta}{1-\alpha-\beta}, \qquad b = \frac{\alpha}{1-\alpha-\beta}.$$

System (3) subject to Neumann boundary conditions and initial values,

$$k_x(0,t) = k_x(\ell,t) = 0, \qquad h_x(0,t) = h_x(\ell,t) = 0, \qquad t > 0,$$
$$k(x,0) = k_0(x), \qquad h(x,0) = h_0(x), \qquad x \in \Omega,$$

with $D_k > 0$ and $D_h > 0$. The parameters D_k and D_h represent the diffusion coefficients for physical capital and human capital, respectively.

The augmented Solow model developed by Mankiw et al. [11] incorporates human capital into the growth framework, while subsequent studies have extended economic growth models by introducing spatial diffusion (e.g., Capasso et al. [13]; Zhong and Huang [14]) and have shown that diffusion can significantly affect stability and generate complex dynamics (e.g., Sutrima et al. [20]). However, these strands of the literature are largely developed in isolation: models with human capital and capital-augmenting technology are predominantly non-spatial, whereas spatial models typically do not incorporate their joint interaction. Consequently, the combined effects of physical capital, human capital, and capital-augmenting technology within a spatial diffusion framework remain insufficiently explored. This gap is important because diffusion mechanisms can fundamentally alter stability properties and produce spatial patterns that cannot be captured by non-spatial models. Therefore, this study develops a spatial Solow model integrating human capital and capital-augmenting technology under diffusion, with a focus on the existence of steady states and the stability of long-term dynamics. The analysis focuses on the dynamic properties of system (3), particularly the existence of steady states and the stability of long-term solutions. This allows us to examine how the interaction between capital accumulation, technological progress and diffusion shapes economic growth dynamics.

From a broader perspective, the results of this study are relevant to sustainable development, particularly in relation to SDG 8 (Decent Work and Economic Growth) and SDG 10 (Reduced Inequalities). The theoretical insights obtained from the stability analysis of system (3) provide an understanding of how capital mobility and human capital distribution influence regional economic balance. In this context, appropriate management of diffusion processes, combined with sustained investment in human capital, can support stable and inclusive economic growth. Furthermore, a more balanced distribution of physical and human capital across regions may help reduce disparities and promote long-term economic sustainability.

2. METHOD

The research methodology used in this study includes a literature review and simulation studies. A literature review involves gathering references from books, journals, or related articles that are relevant to the research questions and objectives [22]. This literature review was conducted by examining references related to dynamic systems theory, stability analysis, reaction-diffusion models for economic growth, and the Solow model. Furthermore, a simulation study involves designing a mathematical model of a system and then conducting computational experiments with that model to describe, explain, and predict the system's dynamics [23]. The analytical component begins by reformulating the Solow model into a system of nonlinear ordinary differential equations. The equilibrium points are determined, and their local stability is analyzed using linearization and eigenvalue analysis based on the Jacobian matrix. Next, the model is extended into a spatial framework by incorporating diffusion terms, resulting in a system of partial differential equations. Linearization around the steady state is performed, leading to a characteristic equation that depends on the spatial mode. Stability is then analyzed using the trace and determinant conditions and dispersion relation to assess the effect of diffusion on the system. Furthermore, conditions for diffusion-driven instability (Turing or Hopf bifurcation) are examined by comparing stability properties in the absence and presence of diffusion. For numerical validation, simulations are carried out using Python or Mathematica. The system is solved under specified initial conditions and parameter values consistent with the theoretical assumptions. A finite difference scheme is used to approximate the spatial derivatives, with appropriate boundary conditions imposed on the domain. Various parameter scenarios are explored to illustrate the dynamic behavior of the system and to confirm the analytical results.

3. RESULTS AND DISCUSSION

This section analyzes the equilibrium and long-term stability of the economic model. Through analytical proofs and numerical simulations, we establish the system's balanced growth path and evaluate the impact of spatial diffusion on capital dynamics. The following subsections detail the transition to an autonomous system, the derivation of steady states, and the system's resilience to perturbations and bifurcations.

3.1. Steady States Profiles and Stability

Before obtaining system (3), a dynamic system was first established from model (1). This dynamic system was formed using the concepts of nonlinear dynamic systems and the transformation (2) per worker, similar to the approach of Szydlowski and Krawiec [24] which was obtained

$$\begin{aligned} \dot{k} &= s_K A(t) k^\alpha h^\beta + (g - \delta - n)k, \\ \dot{h} &= s_H k^\alpha h^\beta - (\delta + n)h. \end{aligned} \tag{4}$$

The nonautonomous dynamic system (4) depends explicitly on the technology $A(t)$ and must be converted into an autonomous system to facilitate steady state analysis. An autonomous system is a continuous dynamic system that does not explicitly depend on time variables [25]. By using the variable transformations $\tilde{k} = kA(t)^{-a}$ and $\tilde{h} = hA(t)^{-b}$, then setting the values of a and b that satisfy the stationarity condition, we can obtain an autonomous dynamic system from the model (1),

$$\begin{aligned} \dot{k} &= s_K k^\alpha h^\beta - (\delta + n + ag)k, \\ \dot{h} &= s_H k^\alpha h^\beta - (\delta + n + bg)h. \end{aligned} \tag{5}$$

where $a = \frac{1-\beta}{1-\alpha-\beta}$ and $b = \frac{\alpha}{1-\alpha-\beta}$.

Let (k^*, h^*)be the steady state to system (3) without diffusion (i.e. with $D_k = D_h = 0$), that is k^*, h^* satisfy $f_1(k^*, h^*) = f_2(k^*, h^*) = 0$. Straightforward computation gives two homogeneous steady states i.e.,

$$S_1(0,0), \qquad S_2\left(\gamma, \gamma \frac{s_H(\delta + n + ag)}{s_K(\delta + n + bg)}\right),$$

where $\gamma = \left[\frac{\delta+n+ag}{s_K}\left(\frac{s_H(\delta+n+ag)}{s_K(\delta+n+\beta g)}\right)^{-\beta}\right]^{\frac{1}{\alpha+\beta-1}}$.

From an economic perspective, steady state S_1 represents a condition in which there is no accumulation of physical or human capital, making it irrelevant for long-term analysis. Steady state S_2, on the other hand, is defined by the condition that $\alpha + \beta < 1$, representing a long-term situation in which investment can offset capital depreciation, population growth, and technological growth. This results in stable levels of physical and human capital.

A stability analysis is performed by linearizing system (5) using the Jacobian matrix around the economically relevant steady state S_2. The Jacobian matrix of system (5) given by

$$J(k^*, h^*) = \begin{bmatrix} (\alpha - 1)(\delta + n + ag) & \beta \dfrac{s_K}{s_H}(\delta + n + bg) \\ \alpha \dfrac{s_H}{s_K}(\delta + n + ag) & (\beta - 1)(\delta + n + bg) \end{bmatrix}. \tag{6}$$

The stability of system (3) without diffusion can be studied from the eigenvalues (λ) of the matrix (6). Matrix J in (6) reduces to a quadratic characteristic equation

$$\lambda^2 - \mathrm{tr}(J)\lambda + \det(J) = 0, \tag{7}$$

where

$$\mathrm{tr}(J) = -[(1-\alpha)(\delta + n + ag) + (1-\beta)(\delta + n + bg)],$$
$$\det(J) = \left((1 - (\alpha + \beta)\right)(\delta + n + ag)\,(\delta + n + bg).$$

We note that the sufficient condition for the steady state (k^*, h^*) is asymptotically stable if all the eigenvalues λ from equation (7) have negative real parts. The basic theory gives that the real parts of the eigenvalues of equation (7) are all negative if and only if $\mathrm{tr}(J) < 0$ and $\det(J) > 0$.

Theorem 3.1.1. The steady state $S_2(k^*, h^*)$ is locally asymptotically stable if and only if

$$\alpha, \beta \in (0,1), \qquad \alpha + \beta < 1. \tag{8}$$

Proof. Based on the assumptions $\alpha, \beta \in (0,1)$, and $\alpha + \beta < 1$, then $\mathrm{tr}(J) < 0$ and $\det(J) > 0$. Therefore, $S_2(k^*, h^*)$ is asymptotically stable. ■

3.2. Stability of System with Diffusion

To include spatial heterogeneity, the model is expanded to include diffusion in physical and human capital. The variables are assumed to depend on both space and time,

$$k = k(x,t), \quad h = h(x,t),$$

thus obtaining a reaction–diffusion system (3). Henceforth, the question of whether diffusion in a spatially distributed system has the potential to destabilize stable steady states will be examined. To analyze this situation, we will consider a small perturbation of the steady states,

$$k(x,t) = k^* + \hat{k}(x,t), \quad h(x,t) = h^* + \hat{h}(x,t), \tag{9}$$

Substituting the perturbation (9) into the system (3), then linearizing the functions f_1, f_2 at (k^*, h^*) using Taylor series, we obtain

$$\hat{k}_t = D_k \hat{k}_{xx} + \left[s_K \alpha (k^*)^{\alpha-1}(h^*)^{\beta} - (\delta + n + ag)\right]\hat{k} + \left[s_K \beta (k^*)^{\alpha}(h^*)^{\beta-1}\right]\hat{h}, \tag{10}$$

$$\hat{h}_t = D_h \hat{h}_{xx} + \left[s_H \alpha (k^*)^{\alpha-1}(h^*)^{\beta}\right]\hat{k} + \left[s_H \beta (k^*)^{\alpha}(h^*)^{\beta-1} - (\delta + n + bg)\right]\hat{h}, \tag{11}$$

$$\hat{k}_x(0,t) = \hat{k}_x(\ell,t) = 0, \quad \hat{k}(x,0) = k_0(x) - k^*, \quad x \in \Omega, \quad t > 0, \tag{12}$$

$$\hat{h}_x(0,t) = \hat{h}_x(\ell,t) = 0, \quad \hat{h}(x,0) = h_0(x) - h^*, \quad x \in \Omega, \quad t > 0. \tag{13}$$

The perturbation function satisfies the linear reaction-diffusion system of equations with Neumann boundary conditions. For the one-dimensional Laplace operator with Neumann boundary conditions, the eigen functions obtained are cosine functions. Therefore, using Fourier series, the perturbations can be written as

$$\hat{k}(x,t) = \sum_{p=0}^{\infty} \kappa_p(t)\cos(px), \qquad \hat{h}(x,t) = \sum_{p=0}^{\infty} \eta_p(t)\cos(px). \tag{14}$$

Here, $p \geq 0$ denotes the general spatial wave number, which represents the frequency of the spatial fluctuations in the perturbation. Then substituting (14) into the linearized system $(10)-(11)$ and canceling out a common factor of $\cos p\,x$, for each p leads to the system

$$\begin{pmatrix} \dot{\kappa}_p(t) \\ \dot{\eta}_p(t) \end{pmatrix} = J_p \begin{pmatrix} \kappa_p(t) \\ \eta_p(t) \end{pmatrix}, \tag{15}$$

where

$$J_p = J(k^*, h^*) - p^2 D, \quad J(k^*, h^*) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad D = \begin{pmatrix} D_k & 0 \\ 0 & D_h \end{pmatrix}, \tag{16}$$

and

$$m_{11} = (\alpha - 1)(\delta + n + ag), \quad m_{12} = \beta \frac{s_K}{s_H}(\delta + n + bg),$$
$$m_{21} = \alpha \frac{s_H}{s_K}(\delta + n + ag), \quad m_{22} = (\beta - 1)(\delta + n + bg).$$

The stability of system (15) without diffusion can be studied from the eigenvalues (λ_p) of the matrix J_p. Matrix J_p in (16) reduces to a quadratic characteristic equation

$$\lambda^2 - \mathrm{tr}(J_p)\lambda + \det(J_p) = 0, \tag{17}$$

where

$$\mathrm{tr}(J_p) = \mathrm{tr}(J) - \mathrm{tr}(D)p^2, \tag{18}$$

$$\det(J_p) = \det(D)\ p^4 - [m_{11}D_h + m_{22}D_k]p^2 + \det(J). \tag{19}$$

Theorem 3.2.1 The steady state (k^*, h^*) of the reaction-diffusion system (3) exhibits local asymptotic stability across all spatial modes $p \geq 0$ under the strict condition that $\alpha + \beta < 1$.
Proof. The stability analysis relies on matrix (16). To establish stability, it must be demonstrated that the trace is negative $(\mathrm{tr}(J_p) < 0)$ and the determinant is positive $(\det(J_p) > 0)$ for every $p \geq 0$. Evaluating these yields the following expressions:
Then, by evaluating (18) and (19) at point S_2, we obtain

$$\mathrm{tr}(J_p) = -[(1-\alpha)(\delta + n + ag) + (1-\beta)(\delta + n + bg)](D_k + D_h)p^2, \tag{21}$$

$$\det(J_p) = (D_k D_h)p^4 + (\delta + n + ag)(\delta + n + bg)\left[\left(1 - (\alpha + \beta)\right) + \left(\frac{D_h(1-\alpha)}{\delta+n+bg} + \frac{D_k(1-\beta)}{\delta+n+ag}\right)p^2\right]. \tag{22}$$

Given the condition $\alpha + \beta < 1$, it logically follows that $\alpha + \beta < 2$. This relationship ensures that $\mathrm{tr}(J_p) < 0$. Furthermore, the initial condition ensures that every coefficient in the determinant's expression is positive, there by guaranteeing that $\det(J_p) > 0$. Consequently, it is proven that the steady state (k^*, h^*) is asymptotically stable for all spatial modes $p \geq 0$ if and only if $\alpha + \beta < 1$. ■

Theorem 3.2.2. If (k^*, h^*) is an asymptotically stable steady state of the system (3) and (k, h) is a solution of the system, then $\lim_{t\to\infty}\left(k(x,t), h(x,t)\right) = (k^*, h^*)$.
Proof. Assuming (14), and solving the homogeneous equations from $(10)-(11)$ using the initial values and boundary conditions from $(12)-(13)$ with the separation of variables and integration factor methods yields the solution

$$\hat{k}(x,t) = \sum_{q=0}^{\infty}\left[w_q(0)e^{-(D_k\lambda_q^2 - m_{11})t} + m_{12}\int_0^t e^{-(D_k\lambda_q^2-m_{11})(t-\tau)}\, z_q(\tau)d\tau\right]\cos(\lambda_q x),$$
$$\hat{h}(x,t) = \sum_{q=0}^{\infty}\left[z_q(0)e^{-(D_h\lambda_q^2 - m_{22})t} + m_{21}\int_0^t e^{-(D_h\lambda_q^2-m_{22})(t-\tau)}\, w_q(\tau)d\tau\right]\cos(\lambda_q x),$$

where

$$w_q(0) = \frac{2}{\ell}\int_0^1 \left(\hat{k}_{0(x)} - k^*\right)\cos(\lambda_q x)\,dx, \quad z_q(0) = \frac{2}{\ell}\int_0^1 \left(\hat{h}_{0(x)} - h^*\right)\cos(\lambda_q x)\,dx,$$

and $\lambda_q = \frac{q\pi}{\ell}$. In this context, the index q $(q = 0,1,2,\dots)$ represents the discrete spatial modes that correspond to the general wave number p used in the previous local stability analysis. The term λ_q defines the specific spatial eigenvalues, ensuring that the eigenfunctions $\cos(\lambda_q x)$ strictly satisfy the Neumann boundary conditions across the specific spatial domain of length l. Although the solutions $w_q(t)$ and $z_q(t)$ contain coupled integral terms involving m_{12} and m_{21}, Theorem 3.2.1 guarantees that the corresponding spatial Jacobian matrix is Hurwitz for all $q \geq 0$ (i.e., its eigenvalues have strictly negative real parts). This structural stability ensures that the exponential decay factors strictly dominate the cross-term interactions, tightly bounding the integrals and forcing them to decay over time. Thus, we have the following limits $\lim_{t\to\infty} w_q(t) = 0$ and $\lim_{t\to\infty} z_q(t) = 0$. Therefore, $\lim_{t\to\infty} \hat{k}(x,t) = 0$ and $\lim_{t\to\infty} \hat{h}(x,t) = 0$. Thus, based on (9), we obtain $\lim_{t\to\infty}\left(k(x,t), h(x,t)\right) = (k^*, h^*)$, so that the equilibrium point S_2 is a locally stable asymptotic steady state in the reaction–diffusion system (3). ■

Economically, this situation demonstrates that the gap in physical and human capital will not persist indefinitely. Even with the diffusion of resources, long-term economic stability remains intact. Since there are no disruptions from wave numbers p, economic fluctuations between regions are only transitory. Thus, capital distribution serves to balance inequalities instead of widening the economic divide between regions.

3.3. Bifurcation Analysis

In this section we will concisely review the bifurcation analysis of system (3). System (3) with steady state S_2 is unstable when $\text{tr}(J_p) \geq 0$ or $\det(J_p) \leq 0$, for some p. In this context, we shall investigate the existence of Turing instability and Hopf bifurcation at the non-trivial steady state i.e. S_2. Based on $\text{tr}(J_p)$ in (18) and (19), if the non-trivial steady state S_2 is asymptotically stable, then diffusion can't affect $\text{tr}(J_p) < 0$. Therefore, the diffusion-driven instability (Turing instability) only depends on the change of sign of the determinant (19). Thus, Turing instability occurs under two conditions: (i) The steady state is stable in the absence of diffusion; (ii) The presence of diffusion destabilizes the steady state ($\det(J_p) < 0$ for some p). On the other hand, Hopf bifurcation occurs under the condition $\text{tr}(J_p) = 0$ and $\det(J_p) > 0$ for some p.

3.3.1 Turing Bifurcation.

Based on Theorem 3.1.1 when $\alpha, \beta \in (0,1)$ and $\alpha + \beta < 1$, we obtain $\text{tr}(J) < 0$ and $\det(J) > 0$. Thus, condition (i) of Turing bifurcation is satisfied. For condition (ii), $\det(J_p)$ represents a quadratic function in p^2, as presented in (19). The extreme point is obtained from the first derivative with respect to p^2,

$$\frac{\partial \det(J_p)}{\partial(p^2)} = 2\det(D)\,p^2 - (m_{11}D_h + m_{22}D_k) = 0,$$

yielding $p_{min}^2 = \frac{m_{11}D_h + m_{22}D_k}{2\det(D)}$. Since $m_{11} < 0, m_{22} < 0, D_k > 0$, and $D_h > 0$, it follows that $p_{min}^2 < 0$. This indicates that the minimum point of $\det(J_p)$ does not lie within the domain $p^2 \geq 0$. Consequently, there exists no wave number p that induces a sign change in the determinant, implying that

$$\det(J_p) > 0 \quad \text{for all } p \geq 0.$$

The theoretical critical threshold for a Turing bifurcation is given by

$$(m_{11}D_h + m_{22}D_k)^2 = 4D_k D_h \det(J),$$

which is derived from the condition of a vanishing discriminant. However, for the economic parameters employed in this model, this condition is not satisfied. Thus, system (3) does not undergo a Turing bifurcation.

3.3.2 Hopf Bifurcation.

A Hopf bifurcation occurs if there exists a value p such that $\mathrm{tr}(J_p) = 0$ and $\det(J_p) > 0$. The expression for the matrix trace yields

$$\mathrm{tr}(J_p) = \mathrm{tr}(J) - (D_k + D_h)p^2.$$

By Theorem 3.1.1, it holds that $\mathrm{tr}(J) < 0$. Since the diffusion coefficients satisfy $D_k + D_h > 0$, it follows that for all $p \geq 0, \mathrm{tr}(J_p) < 0$. Consequently, the condition $\mathrm{tr}(J_p) = 0$ cannot be satisfied, neither for $p = 0$, which corresponds to a temporal Hopf bifurcation, nor for $p \neq 0$, which corresponds to a spatial Hopf bifurcation. Therefore, a Hopf bifurcation does not occur in system (3).

The results obtained in this study are consistent with classical findings in spatial economic growth models, particularly those suggesting that diffusion tends to reduce regional disparities rather than amplify them. In contrast to certain reaction–diffusion systems in biology, where diffusion can induce Turing bifurcation and Hopf bifurcation, the present model demonstrates that economic diffusion functions as a stabilising agent under diminishing returns.

3.4. Numerical Simulation

The present section offers numerical simulations with a view to validating the theoretical findings regarding the stability of the spatial Solow model. In order to guarantee strict reproducibility, the initial value problems are solved using Python for numerical integration, while Wolfram Mathematica is utilised for phase portrait visualisation. The baseline economic parameters are established as follows: production elasticities $\alpha = 0.3$ and $\beta = 0.4$, investment rates $s_K = 0.3$ and $s_H = 0.25$, depreciation rate $\delta = 0.05$, and exogenous growth rates $n = 0.02$ and $g = 0.02$. The spatial diffusion coefficients for physical and human capital are set to $D_k = 0.01$ and $D_h = 0.03$, respectively. Under these parameters, the economically relevant steady state evaluates to $S_2(k^*, h^*) = (29.04, 29.58)$.

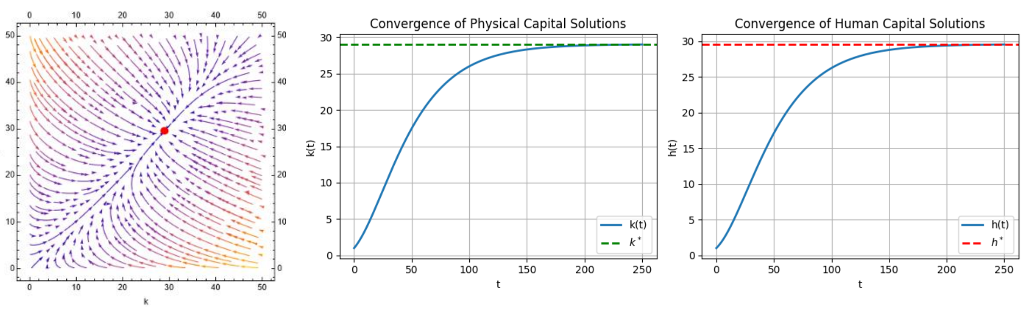
The nonspatial system corresponding to equation (3)without diffusion is solved using the classical fourth order Runge–Kutta (RK4) method in Python. The time step is set to $\Delta t = 0.01$ over the interval $t \in [0, 250]$, with initial condition $(k(0), h(0)) = (1,1)$. In addition, the phase portrait is constructed in Wolfram Mathematica using a stream plot representation of the vector field, combined with numerical trajectories obtained via NDSolve. Multiple initial conditions are selected across the state space to verify that all trajectories converge toward the steady state $S_2(k^*, h^*)$. This provides numerical confirmation of the local asymptotic stability established in Theorem 3.1.1 and is consistent with the stability characterization given in Theorem 3.2.1.

For the spatial model, the reaction–diffusion system is discretized on a one-dimensional domain $x \in [0,100]$. A second-order central finite difference scheme is used to approximate the Laplacian operator, and time integration is carried out using an explicit forward Euler method implemented in Python. Homogeneous Neumann boundary conditions are imposed. The discretization parameters are $N_x = 120$, $\Delta x = \frac{100}{119}$, $\Delta t = 5 \times 10^{-4}$, and final time $T = 150$, chosen to satisfy the standard stability condition for explicit diffusion schemes. To represent spatial heterogeneity, the initial conditions are defined as small perturbations around the steady state,

$$k(x,0) = k^*\big(1 + 0.01, \xi_1(x)\big), \;\; h(x,0) = h^*\big(1 + 0.01, \xi_2(x)\big),$$

where $\xi_1(x)$ and $\xi_2(x)$ are uniformly distributed random variables in [0,1], generated using a fixed random seed.

The dynamics approaching the steady state in Figure 1 are illustrated via a phase plane depicting the system's trajectories within the state space (k, h), as well as the convergence of the physical and human capital solutions

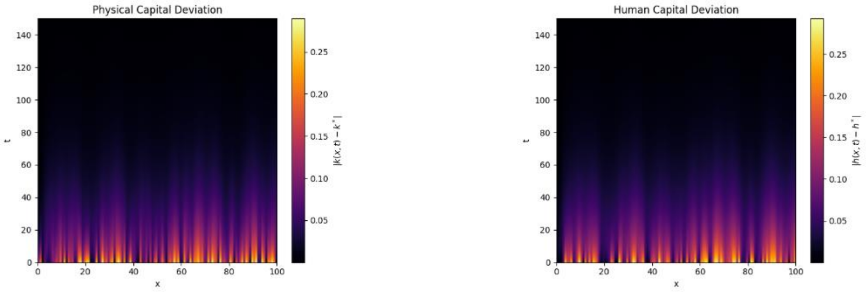


(a) (b)

Figure 1. (a) Phase plane of the steady state S_2 with the red dot represent $S_2(29.04, 29.58)$ for $\alpha = 0.3, \beta = 0.4, s_K = 0.3, s_H = 0.25, \delta = 0.05, n = 0.02, g = 0.02$; (b) Convergence of the solution $k(t)$ and $h(t)$ for $\alpha = 0.3, \beta = 0.4, s_K = 0.3, s_H = 0.25, \delta = 0.05, n = 0.02, g = 0.02, (k^*, h^*) = (29.04, 29.58)$.

The phase plane in Figure 1(a) displays the direction of the system's movement within the state space (k, h). All trajectories originating from various initial conditions move towards the long-run steady state $S_2(29.04, 29.58)$. This indicates that point S_2 represents a stable steady state, consistent with the satisfaction of the asymptotic stability conditions. Furthermore, Figure 1(b) demonstrate the convergence of the temporal dynamics of physical and human capital, which increase from their initial conditions toward their steady states. Growth proceeds rapidly in the initial stages due to high investment productivity, then gradually slows as diminishing returns set in. In economic terms, this condition illustrates the natural adjustment mechanism within the economy, where investment, capital depreciation, labor force growth, and technological progress interact to steer the system toward a long run steady state.

Furthermore, the stability of the system with diffusion is guaranteed by Theorem 3.2.1. The theorem states that the steady state (k^*, h^*) exhibits local asymptotic stability across all spatial modes if the strict condition $\alpha + \beta < 1$ is met. Substituting the given parameter values ($\alpha = 0.3$ and $\beta = 0.4$), we obtain $\alpha + \beta = 0.7$. Since $0.7 < 1$, the conditions of Theorem 3.2.1 are completely satisfied.



(a) (b)

Figure 2. (a) Spatial deviation of physical capital from the steady state $S_2(k^*, h^*) = (29.04, 29.58)$ in system (3) for $\alpha = 0.3, \beta = 0.4, s_K = 0.3, s_H = 0.25, \delta = 0.05, n = 0.02, g = 0.02, D_k = 0.01$; (b) Spatial deviation of human capital from the steady state $(k^*, h^*) = (29.04, 29.58)$ in system (3) for $\alpha = 0.3, \beta = 0.4, s_K = 0.3, s_H = 0.25, \delta = 0.05, n = 0.02, g = 0.02, D_h = 0.03$.

From an economic perspective, Figure 2 illustrates that, during the initial stages of development, regional disparities persist due to variations in the initial conditions of physical and human capital. However, over time, diffusion facilitates the dispersion of capital and knowledge, there by narrowing these disparities as all regions gradually converge toward the long-run equilibrium. This indicates that inter-regional interaction fosters the equalization process without compromising the overall stability of economic growth.

4. CONCLUSION

A dynamical systems approach is employed to analyze a spatial Solow model incorporating capital-augmenting technology and human capital. Consistent with the initial hypothesis regarding regional disparities, the analysis reveals that the system possesses an asymptotically stable steady state, provided that the production elasticities satisfy the condition of diminishing returns. Upon the introduction of diffusion, our analysis demonstrates that the stability of this steady state is unconditionally preserved across all spatial modes ($p \geq 0$). Neither Turing nor Hopf bifurcations occur within the system (3). Consequently, diffusion induces neither a change in stability nor the emergence of novel spatial patterns. Overall, the system exhibits robust convergence toward a long-run steady state.From an economic and applied perspective, these findings indicate that regional disparities in physical and human capital are transitory. Spatial diffusion acts as a natural equalization mechanism, driving the entire economic system toward synchronized, stable long-run growth. Because this convergence does not depend on a restrictive diffusion-ratio threshold, policymakers can confidently formulate targeted investments, especially in human capital development, knowing that the spatial mobility of resources will inherently support inclusive and balanced regional growth. Looking forward, this framework can be expanded to model more complex spatial dynamics using time delays or fractional derivatives.

Overall, the findings highlight the important role of human capital and technological efficiency in maintaining long-term economic stability. The incorporation of spatial effects lends further credence to the hypothesis that regional interactions can facilitate convergence processes without engendering instability. These results provide a theoretical foundation for emphasizing policies that support human capital development and the diffusion of technology across regions.

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