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Stability of the Solow Model with Hicks-Neutral Technology under Capital and Human Capital Diffusion

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Abstract: *Differences in economic growth across regions indicate that spatial interactions between physical capital and human capital may influence long-run economic dynamics. To investigate these spatial interactions, this study develops a reaction-diffusion-based Solow model incorporating Hicks-neutral technology and human capital to analyze the stability of its dynamic system. Through a detrending process, the system becomes autonomous and is expressed in terms variables per effective worker, thereby enabling a steady-state stability analysis related to a balanced growth path. Analysis results show that without diffusion, the steady state is locally stable as long as conditions of decreasing returns to scale hold. When spatial diffusion with homogeneous Neumann boundary conditions is introduced, results from linearity and eigenvalue analysis indicate that the real part of all eigenvalues remains negative for all spatial modes. This implies that diffusion does not trigger either Hopf or Turing bifurcations, so no persistent spatial patterns form. Numerical simulations confirm the analytical results and illustrate the exponential decay of deviations toward the balanced growth path. These results suggest that the system remains stable despite modal diffusion and highlight the application of reaction–diffusion modeling and dynamical systems analysis as mathematical tools for studying spatial economic growth.*
Keywords: *Solow model, reaction-diffusion system, steady state, stability analysis, bifurcation*

1. INTRODUCTION

Economic growth is one of the primary indicators of a country's long-run economic performance and is commonly measured by Gross Domestic Product (GDP). However, it often exhibits regional disparities due to differences in savings or investment rates, population growth, technological progress, and human capital quality [1], [2]. Understanding how these factors interact is therefore important for explaining long-run economic dynamics. The neoclassical growth model introduced by Solow [3] provides a fundamental theoretical basis for examining long-run growth through capital accumulation, labor, and exogenous technological progress. This model was later extended by Mankiw et al. [4], to include human capital as a production factor. One concept of technological progress is Hicks-neutral technology, which proportionally increases the productivity of all production factors without altering the substitution ratio between capital and labor [5]. Under these assumptions, the production function is given by the Cobb–Douglas form

$$Y(t) = A(t)K(t)^{\alpha}H(t)^{\beta}L(t)^{1-\alpha-\beta},$$

where $Y(t)$, $A(t)$, $K(t)$, $H(t)$, and $L(t)$ denote total output, technology, physical capital, human capital, and labor, respectively. The parameters α and β represent the output elasticities of physical and human capital, with the conditions $0 < \alpha, \beta < 1$ and $\alpha + \beta < 1$, ensuring decreasing returns to scale. Technology and labor are assumed to grow exogenously at constant rates g and n, respectively. The model can be interpreted as an economic dynamical system, which possesses a unique steady state representing the balanced growth path and is locally stable in the ordinary differential equation formulation under suitable assumptions [6], [7].

The classical Solow model does not explicitly consider spatial interactions between regions, although spatial factors and the quality of human capital may influence productivity growth across regions [8], [9]. To address this limitation, several studies have extended economic growth models to a spatial framework. For instance, González–Parra et al. [10] showed that physical capital diffusion in a spatial Solow model affects long-run dynamics and spatial distribution. In reaction–diffusion systems, diffusion may alter the stability properties and potentially generate spatial patterns through mechanisms such as Turing bifurcation [11]. In particular, Turing bifurcation may arise when the diffusion rates between interacting variables differ significantly [12]. Furthermore, diffusion-driven mechanisms may also produce complex spatiotemporal patterns depending on the interaction between nonlinear reaction terms and diffusion processes [13], [14]. In economics, several studies have shown that the diffusion of key economic variables, such as capital, labor, or employment, can influence the stability of steady states and the emergence of spatial patterns [15], [16], [17]. Furthermore, Sutrima et al. [18] analyze changes in stability and bifurcation induced by diffusion. However, other studies show that diffusion does not necessarily lead to instability, since systems with a stable homogeneous steady state may remain stable for all spatial modes [19], [20]. Therefore, stability and bifurcation analysis are fundamental steps in the development of spatial economic growth models.

Although several spatial extensions of the Solow model have been proposed, most existing studies focus primarily on the diffusion of physical capital and pay limited attention to the role of human capital and Hicks–neutral technology in a spatial framework. As a result, the stability properties of spatial Solow models that simultaneously incorporate the diffusion of physical and human capital together with Hicks–neutral technology remain relatively less explored in the literature. This gap has clear analytical relevance, from the perspective of sustainable development, whether capital diffusion stabilizes or destabilizes regional economies, in which case diffusion tends to equalize or magnify inequalities. Answering this question analytically and verifying it under predetermined parameters distinguishes this study from previous work that ignores human capital or only considers diffusion in physical capital. Motivated by these considerations, this study investigates the dynamical system of the Solow model with Hicks–neutral technology and human capital in a spatial framework. The model is formulated as a reaction–diffusion system describing the spatial dynamics of physical and human capital.

$$\begin{aligned} u_t(x,t) &= s_K\, u(x,t)^{\alpha}\, v(x,t)^{\beta} - \mu\, u(x,t) + D_u\, u_{xx}(x,t),\\ v_t(x,t) &= s_H\, u(x,t)^{\alpha}\, v(x,t)^{\beta} - \mu\, v(x,t) + D_v\, v_{xx}(x,t), \end{aligned} \tag{1}$$

where $D_u,\ D_v > 0$, equipped with homogeneous Neumann boundary conditions,

$$\begin{aligned} u_x(0,t) = u_x(L,t) = 0, \quad & v_x(0,t) = v_x(L,t) = 0, \qquad t > 0,\\ u(x,0) = u_0(x), \qquad & v(x,0) = v_0(x), \qquad x \in [0,L]. \end{aligned} \tag{2}$$

Here, u and v represent physical and human capital per effective worker after the detrending. The parameters s_K and s_H represent the savings rates allocated to the accumulation of physical and human capital. The effective depreciation rate is given by $\mu = \delta + n + \theta g$, where $\delta, n,$ and g denote the depreciation rate, the population growth rate, and the rate of technological progress, respectively. The scalar, $\theta = \frac{1}{1-\alpha-\beta}$ arises from the detrending transformation used to eliminate the explicit dependence on technological growth so that the system becomes autonomous. The coefficients D_u and D_v represent the diffusion rates of physical and human capital, while the terms u_{xx} and v_{xx} model one-dimensional spatial diffusion. Neumann boundary conditions are used to represent zero–flux boundaries corresponding to a closed economic system [21].

The main objective of this research is to analyze the stability of the steady state representing the balanced growth path and to examine whether diffusion can induce bifurcations that generate spatial patterns. To achieve these objectives, the stability of the system is examined using dynamical systems techniques, including linearization, eigenvalue analysis, and bifurcation criteria [22], [23], [24]. Numerical simulations are also provided to illustrate the analytical results.

2. METHOD

This study utilizes a literature review and simulation approach within a mathematical modeling context to examine the stability characteristics of the spatial Solow model. The study's theoretical framework is established through a review of pertinent literature on economic growth theory, the Solow model, dynamical systems, reaction-diffusion modeling, and bifurcation. In this study, the dynamical system is formulated based on the Cobb–Douglas production function, and a detrending process is applied to eliminate the effects of exogenous technological and labor growth, resulting in an autonomous system [25]. To convert them into an autonomous form, physical and human capital per effective worker are defined as $u(t) = \frac{k(t)}{A(t)^{\theta}}$ and $v(t) = \frac{h(t)}{A(t)^{\theta}}$, where $\theta = \frac{1}{1-\alpha-\beta}$. Here $k = \frac{K}{L}$ and $h = \frac{H}{L}$ are capital stocks per worker, and $A(t)^{\theta}$ rescales for technology. Substituting these into the capital accumulation equations and using $\mu = \delta + n + \theta g$ as the effective depreciation rate yields the autonomous system (1), expressed entirely in per-effective-worker variables. The system is then analyzed by linearizing the system and examining the eigenvalues of the Jacobian matrix to determine the stability of the balanced growth path and to investigate the effect of diffusion on the system dynamics. The overall research procedure is shown in Figure 1.

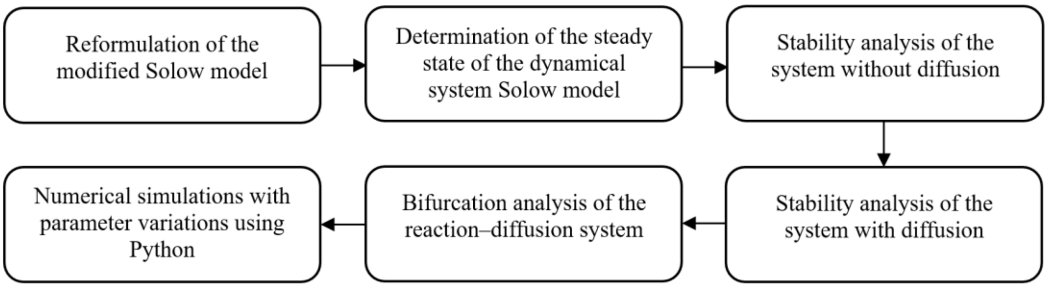


Figure 1. Research procedure

3. RESULTS AND DISCUSSION

3.1. Steady State of the Dynamical System Solow Model

The steady state of the Solow dynamical system with Hicks–neutral technology and human capital is determined from system (1) without diffusion $(D_u = D_v = 0)$, so that the system is only influenced by the reaction function, $f(u,v) = s_K u^{\alpha} v^{\beta} - \mu\, u$ and $g(u,v) = s_H u^{\alpha} v^{\beta} - \mu\, v$. The steady state (u^*, v^*) is determined by the conditions $f(u^*, v^*) = g(u^*, v^*) = 0$, so

$$\begin{aligned} f(u^*, v^*) &= s_K (u^*)^{\alpha} (v^*)^{\beta} - \mu\, u^* = 0, \\ g(u^*, v^*) &= s_H (u^*)^{\alpha} (v^*)^{\beta} - \mu\, v^* = 0. \end{aligned} \tag{3}$$

System (3) has two types of steady states, the trivial steady state $(0,0)$, which is not economically meaningful, and a nontrivial steady state. From (3), we obtain the relationship

$$u^* = \frac{s_K}{s_H} v^*. \tag{4}$$

Substituting (4) into the system (3), we obtain

$$u^* = \left(\frac{{s_K}^{1-\beta} {s_H}^{\beta}}{\mu}\right)^{\frac{1}{1-\alpha-\beta}}, \qquad v^* = \left(\frac{{s_H}^{1-\alpha} {s_K}^{\alpha}}{\mu}\right)^{\frac{1}{1-\alpha-\beta}}. \tag{5}$$

where $s_K, s_H, \mu > 0$ and $\alpha + \beta < 1$. Economically, the nontrivial steady state (u^*, v^*) indicates a balanced growth path in which physical capital and human capital per effective worker remain constant in the long run.

3.2. Stability Analysis without Diffusion

The stability of the steady state for the system (1) without diffusion is analyzed by linearizing $f(u,v)$ and $g(u,v)$, yielding the Jacobian matrix,

$$J(u,v) = \begin{bmatrix} \alpha s_K u^{\alpha-1} v^{\beta} - \mu & \beta s_K u^{\alpha} v^{\beta-1} \\ \alpha s_H u^{\alpha-1} v^{\beta} & \beta s_H u^{\alpha} v^{\beta-1} - \mu \end{bmatrix}. \tag{6}$$

The stability property of the system is summarized in the following theorem.

Theorem 3.1 *The steady state (u^*, v^*) of the system (1) without diffusion is locally asymptotically stable if and only if $\alpha + \beta < 1$.*

Proof. By evaluating the Jacobian matrix (6) at the steady state (5), we obtain

$$J^* = \begin{bmatrix} (\alpha-1)\mu & \beta\mu\dfrac{s_K}{s_H} \\ \alpha\mu\dfrac{s_H}{s_K} & (\beta-1)\mu \end{bmatrix}. \tag{7}$$

Based on the stability criteria [23], the steady state is locally asymptotically stable if all eigenvalues of the characteristic equation have negative real parts, $(\Re(\lambda) < 0)$. The eigenvalues are determined by the characteristic equation of the matrix (7),

$$\lambda^2 - \mathrm{tr}(J^*)\lambda + \det(J^*) = 0,$$

where

$$\mathrm{tr}(J^*) = (\alpha+\beta-2)\mu, \qquad \text{and} \qquad \det(J^*) = (1-\alpha-\beta)\mu^2.$$

Since $\mu > 0$, the determinant of the matrix J^* (7) is always positive, $\det(J^*) > 0$ if and only if

$$1-\alpha-\beta > 0,$$

which is equivalent to $\alpha+\beta < 1$. Therefore, the steady state (u^*, v^*) is asymptotically stable when the trace of the Jacobian matrix (7) is negative, $\mathrm{tr}(J^*) < 0$, that is, when

$$\alpha+\beta < 1. \qquad \square$$

Economically, with decreasing returns to reproducible factors, the marginal product of both physical and human capital declines as accumulation increases. Investment therefore runs into diminishing returns before it can drive capital stocks to infinity, and the net accumulation rate eventually falls to zero at the balanced growth path. This mechanism is based on decreasing returns, which Theorem 3.1 formalizes mathematically.

3.3. Stability Analysis with Diffusion

The reaction–diffusion system (1) is defined on the domain $\Omega = (0, L)$ for $t > 0$, considering the spatial diffusion of physical and human capital under Neumann boundary conditions. Economically, this condition represents a closed system, in which factor mobility occurs only within the domain and no external inflow or outflow is allowed. Then, to analyze the stability of the reaction–diffusion system in a neighborhood of the steady state, linearization is performed with small perturbations to the steady state (u^*, v^*),

$$u(x,t) = u^* + a(x,t), \qquad v(x,t) = v^* + b(x,t), \tag{8}$$

where $|a(x,t)|, |b(x,t)| \ll 1$. This assumption allows a linear approximation via a first-order Taylor expansion around (u^*, v^*), with $f(u^*, v^*) = g(u^*, v^*) = 0$. Substituting (8) into system (1) yields the linearized system,

$$\begin{aligned} a_t(x,t) &= \left(s_K\alpha(u^*)^{\alpha-1}(v^*)^{\beta} - \mu\right)a(x,t) + \left(s_K\beta(u^*)^{\alpha}(v^*)^{\beta-1}\right)b(x,t) + D_u a_{xx}(x,t), \\ b_t(x,t) &= \left(s_H\alpha(u^*)^{\alpha-1}(v^*)^{\beta}\right)a(x,t) + \left(s_H\beta(u^*)^{\alpha}(v^*)^{\beta-1} - \mu\right)b(x,t) + D_v b_{xx}(x,t), \end{aligned} \tag{9}$$

with Neumann homogeneous boundary conditions and initial values,

$$\begin{aligned} &a_x(0,t) = a_x(L,t) = 0, \qquad b_x(0,t) = b_x(L,t) = 0, \qquad t > 0, \\ &a(x,0) = u_0(x) - u^*, \qquad b(x,0) = v_0(x) - v^*, \qquad x \in [0, L]. \end{aligned} \tag{10}$$

Each perturbation can be expanded as a Fourier series into cosine modes that satisfy the Neumann boundary conditions.

$$a(x,t) = \sum_{m=0}^{\infty} \xi_m(t)\cos(mx), \qquad b(x,t) = \sum_{m=0}^{\infty} \eta_m(t)\cos(mx), \tag{11}$$

where m denotes the wave number, when $m = 0$ indicates homogeneous perturbation, while $m \neq 0$ shows heterogeneous perturbation that causes spatial variation.

Substituting (11) into (9) leads, for each mode m,

$$\begin{bmatrix} \dot{\xi}_m(t) \\ \dot{\eta}_m(t) \end{bmatrix} = J_m \begin{bmatrix} \xi_m(t) \\ \eta_m(t) \end{bmatrix}, \tag{12}$$

where

$$J_m := J - Dm^2, \quad J = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad D = \begin{bmatrix} D_u & 0 \\ 0 & D_v \end{bmatrix},$$

and

$$\begin{aligned} a_{11} &= s_K\alpha(u^*)^{\alpha-1}(v^*)^{\beta} - \mu, \quad & a_{12} &= s_K\beta(u^*)^{\alpha}(v^*)^{\beta-1}, \\ a_{21} &= s_H\alpha(u^*)^{\alpha-1}(v^*)^{\beta}, \quad & a_{22} &= s_H\beta(u^*)^{\alpha}(v^*)^{\beta-1} - \mu, \end{aligned}$$

so that for all spatial mode m, the stability of the system is determined by the matrix J_m,

$$J_m = \begin{bmatrix} \left(s_K\alpha(u^*)^{\alpha-1}(v^*)^{\beta} - \mu\right) - D_u m^2 & \left(s_K\beta(u^*)^{\alpha}(v^*)^{\beta-1}\right) \\ \left(s_H\alpha(u^*)^{\alpha-1}(v^*)^{\beta}\right) & \left(s_H\beta(u^*)^{\alpha}(v^*)^{\beta-1} - \mu\right) - D_v m^2 \end{bmatrix} = \begin{bmatrix} (\alpha-1)\mu - D_u m^2 & \beta\mu\,\frac{s_K}{s_H} \\ \alpha\mu\,\frac{s_H}{s_K} & (\beta-1)\mu - D_v m^2 \end{bmatrix}. \tag{13}$$

The stability of the system (9) can be examined through the eigenvalues of J_m, which satisfy the characteristic equation

$$\lambda^2 - \operatorname{tr}(J_m)\lambda + \det(J_m) = 0,$$

where

$$\begin{aligned} \operatorname{tr}(J_m) &= \operatorname{tr}(J) - \operatorname{tr}(D)m^2, \\ \det(J_m) &= \det(J) - (a_{11}D_v + a_{22}D_u)m^2 + \det(D)\,m^4. \end{aligned} \tag{14}$$

Theorem 3.2 *The reaction–diffusion system (9) is locally asymptotically stable at the steady state (u^*, v^*) for all spatial modes $m \geq 0$ if and only if $\alpha + \beta < 1$.*

Proof. Stability is analyzed using matrix (13), and it will be proven that $\operatorname{tr}(J_m) < 0$ and $\det(J_m) > 0$ for all $m \geq 0$, we obtain

$$\begin{aligned} \operatorname{tr}(J_m) &= (\alpha + \beta - 2)\mu - (D_u + D_v)m^2, \\ \det(J_m) &= (1 - \alpha - \beta)\mu^2 + [(1-\alpha)D_v + (1-\beta)D_u]\mu m^2 + D_u D_v m^4. \end{aligned} \tag{15}$$

If $\alpha + \beta < 1$, then $\alpha + \beta < 2$ holds, so that $\operatorname{tr}(J_m) < 0$ and all coefficients in the determinant are positive, which guarantees that $\det(J_m) > 0$. Thus, the steady state (u^*, v^*) is asymptotically stable for all spatial modes $m \geq 0$ if and only if

$$\alpha + \beta < 1. \qquad \square$$

Economically, the matrix J_m represents the interaction between capital accumulation and the distribution of production factors. The negative diffusion term in the trace J_m means that higher spatial modes decay faster than the homogeneous mode $m = 0$, so that diffusion accelerates the damping of regional capital differences rather than amplifying them. Therefore, interregional capital disparities are transient. Thus, the solution converges uniformly to the steady state (u^*, v^*), as stated in the following theorem.

Theorem 3.3 *If (u^*, v^*) and $\left(u(x,t), v(x,t)\right)$ are, respectively, the asymptotically stable steady state of the reaction–diffusion system (1) with Neumann boundary conditions (2) and the solution of that system, then $\lim_{t\to\infty}\left(u(x,t), v(x,t)\right) = (u^*, v^*)$, uniformly for every $x \in [0, L]$.*

Proof. Set $a(x,t) = u(x,t) - u^*$ and $b(x,t) = v(x,t) - v^*$, These measure pointwise deviations from the steady state and satisfy the linearized system (9) with boundary conditions (10). The one-dimensional Laplace operator on $(0, L)$, with Neumann boundary conditions has eigenfunctions,

$$\phi_n(x) = \cos(\lambda_n x), \qquad \lambda_n = \frac{n\pi}{L},\ n = 0,1,2,\ldots,$$

corresponding to eigenvalues λ_n^2, form an orthogonal basis of $L^2(0, L)$. The deviations admit the cosine Fourier expansions,

$$a(x,t) = \sum_{n=0}^{\infty} \xi_n(t)\cos(\lambda_n x), \qquad b(x,t) = \sum_{n=0}^{\infty} \eta_n(t)\cos(\lambda_n x). \tag{16}$$

Then, by substituting into the system (9), we obtain

$$\sum_{n=0}^{\infty}\left[\dot{\xi}_n(t) - (a_{11} - D_u\lambda_n^2)\xi_n(t) - a_{12}\eta_n(t)\right]\cos(\lambda_n x) = 0,$$

$$\sum_{n=0}^{\infty}\left[\dot{\eta}_n(t) - (a_{22} - D_v\lambda_n^2)\eta_n(t) - a_{21}\xi_n(t)\right]\cos(\lambda_n x) = 0.$$

Each equation is multiplied by $\cos(\lambda_m x)$ and integrated over $x \in (0, L)$, and using the orthogonality of eigenfunctions, we obtain for each $n \geq 0$ the system of ODE,

$$\begin{aligned} \dot{\xi}_n(t) - (a_{11} - D_u\lambda_n^2)\xi_n(t) &= a_{12}\eta_n(t), \\ \dot{\eta}_n(t) - (a_{22} - D_v\lambda_n^2)\eta_n(t) &= a_{21}\xi_n(t). \end{aligned} \tag{17}$$

System (17) is a first-order linear system of ODE, with the integral factors $I_n^{(\xi)}(t) = e^{-(a_{11}-D_u\lambda_n^2)t}$ and $I_n^{(\eta)}(t) = e^{-(a_{22}-D_v\lambda_n^2)t}$, so that

$$\begin{aligned} \frac{d}{dt}\left(e^{-(a_{11}-D_u\lambda_n^2)t}\,\xi_n(t)\right) &= a_{12}\,e^{-(a_{11}-D_u\lambda_n^2)t}\,\eta_n(t), \\ \frac{d}{dt}\left(e^{-(a_{22}-D_v\lambda_n^2)t}\,\eta_n(t)\right) &= a_{21}\,e^{-(a_{22}-D_v\lambda_n^2)t}\,\xi_n(t), \end{aligned} \tag{18}$$

by integrating (18) with respect to t, yields

$$\begin{aligned} \xi_n(t) &= e^{(a_{11}-D_u\lambda_n^2)t}\xi_n(0) + a_{12}\int_0^t e^{(a_{11}-D_u\lambda_n^2)(t-s)}\,\eta_n(s)\,ds, \\ \eta_n(t) &= e^{(a_{22}-D_v\lambda_n^2)t}\eta_n(0) + a_{21}\int_0^t e^{(a_{22}-D_v\lambda_n^2)(t-s)}\,\xi_n(s)\,ds. \end{aligned} \tag{19}$$

From the Jacobian matrix (7), the diagonal entries at the steady state are $a_{11} = (\alpha - 1)\mu$ and $a_{22} = (\beta - 1)\mu$. Since $0 < \alpha < 1$, $0 < \beta < 1$ and $\mu > 0$, it follows that $a_{11} < 0$ and $a_{22} < 0$. Consequently, $a_{11} - D_u\lambda_n^2 < 0$ and $a_{22} - D_v\lambda_n^2 < 0$ for every $n \geq 0$. Therefore, the first term in the solution $\xi_n(t)$ decays exponentially and for the integral term, the factor $e^{(a_{11}-D_u\lambda_n^2)(t-s)} \to 0$ as $t \to \infty$. By the same argument for $\eta_n(t)$, it follows that

$$\lim_{t\to\infty} \xi_n(t) = 0, \qquad \lim_{t\to\infty} \eta_n(t) = 0.$$

Since the eigenfunctions $\cos(\lambda_n x)$ are bounded on $(0, L)$ and the Fourier coefficients $\xi_n(t)$ and $\eta_n(t)$ decay exponentially in time, the Fourier series converges uniformly in space. Therefore,

$$\lim_{t\to\infty} a\,(x,t) = 0, \qquad \lim_{t\to\infty} b\,(x,t) = 0,$$

uniformly for all $x \in [0, L]$. Thus, the solution of the reaction–diffusion system converges to a homogeneous steady state,

$$\lim_{t\to\infty}\big(u(x,t), v(x,t)\big) = (u^*, v^*). \qquad \square$$

The uniform convergence established here has a direct economic reading, any spatial disparity in physical or human capital across locations is temporary. Regions that start above or below the balanced growth path will each return to it simultaneously. In this sense, Theorem 3.3 formalizes the idea of spatial convergence within the closed regional economy.

3.4. Bifurcation Analysis

Bifurcation analysis is conducted to examine the possibility of changes in the stability of the system due to parameter variations, particularly due to the addition of diffusion effects. This analysis focuses on the nontrivial steady state (u^*, v^*), which represents a balanced growth path and has been proven to be asymptotically stable in systems without diffusion. In reaction–diffusion systems, bifurcation is characterized by

$$\mathrm{tr}(J_m) \geq 0 \text{ or } \det(J_m) \leq 0.$$

A Hopf bifurcation occurs when a pair of complex conjugate eigenvalues crosses the imaginary axis, resulting in periodic oscillations around the steady state. A necessary condition for the occurrence of a Hopf bifurcation is given by

$$\mathrm{tr}(J_m) = 0 \quad \text{and} \quad \det(J_m) > 0. \tag{20}$$

However, since $\alpha + \beta < 1$ and $\mu > 0$, it follows that $(\alpha + \beta - 2)\mu < 0$. Furthermore, $(D_u + D_v)m^2 \geq 0$ for every $m \geq 0$. Thus, the addition of diffusion does not change the sign of $\mathrm{tr}(J_m)$ for all m, because $\mathrm{tr}(J_m)$ is always negative. Therefore, condition (20) can never be satisfied, so it can be concluded that a Hopf bifurcation does not occur in this system.

Furthermore, a Turing bifurcation may occur when a stable steady state becomes unstable to spatial perturbations, that is, for some $m \neq 0$. In this case, since the trace of the Jacobian matrix is always negative, Turing instability is determined by the sign of the determinant, so the condition for Turing instability is given by

$$\det(J_m) < 0. \tag{21}$$

The critical condition of the Turing bifurcation is reached when $\det(J_m) = 0$ for some value of m. This marks the boundary between stable conditions and the emergence of instability. Furthermore, $\det(J_m)$ can be regarded as a quadratic function of m^2,

$$\det(J_m) = D_u D_v m^4 + \left[\left((1-\alpha)D_v + (1-\beta)D_u\right)\mu\right]m^2 + (1-\alpha-\beta)\mu^2, \tag{22}$$

An examination of the quadratic equation $\det(J_m) = 0$ shows that, under the condition $\alpha + \beta < 1$, all coefficients are positive. Consequently, there are no positive real roots, and this yields $m^2 < 0$, which lies outside the admissible domain $m \geq 0$. Therefore, the minimum value of the determinant is attained at the boundary of the domain when $m = 0$, and we obtain

$$\det(J_m)_{\min} = (1-\alpha-\beta)\mu^2. \tag{23}$$

Note that the case $m = 0$ corresponds to the spatially homogeneous mode, which represents the system without diffusion effects. Since $\det(J_m) > 0$ for all $m \geq 0$, the condition for Turing bifurcation (21) is not satisfied, and the critical condition for Turing bifurcation is never reached. Based on the analysis of $\mathrm{tr}(J_m)$ and $\det(J_m)$, the following theorem is proven.

Theorem 3.4 *The reaction–diffusion Solow model with Hicks–neutral technology and human capital does not experience either Hopf bifurcation or Turing bifurcation for any parameter values that satisfy*

$$\alpha + \beta < 1, \quad \mu > 0, \quad D_u > 0, \quad D_v > 0.$$

3.5. Numerical Simulations

Numerical simulations are performed to validate the analytical results. Developing countries often face the problem of regional growth disparities. Differences in the accumulation of physical capital, the quality of human capital, and access to technology cause regions with stronger productive capacity to grow faster than others. The production structure shows characteristics of decreasing returns to scale, with output elasticities of physical capital and human capital, respectively, given by $\alpha = 0.3$ and $\beta = 0.25$, so that $\alpha + \beta < 1$. On the other hand, the country has allocated a significant portion of investment, i.e., $s_K = 0.35$ for physical capital and $s_H = 0.15$ for human capital. Next, the effective capital loss rate is based on the parameters $\delta = 0.05$, $n = 0.015$, and $g = 0.03$, so the effective capital loss rate is calculated as

$$\mu = \delta + n + \theta g = 0.05 + 0.015 + 2.222 \times 0.03 \approx 0.1317.$$

This value indicates that, in addition to depreciation and population growth, Hicks-neutral technology makes a significant contribution to the dynamics of capital accumulation.

The simulations are implemented in Python using a finite difference discretization of the system. Temporal discretization is carried out using an explicit forward Euler scheme, while spatial derivatives are approximated by central differences. The numerical scheme is given by

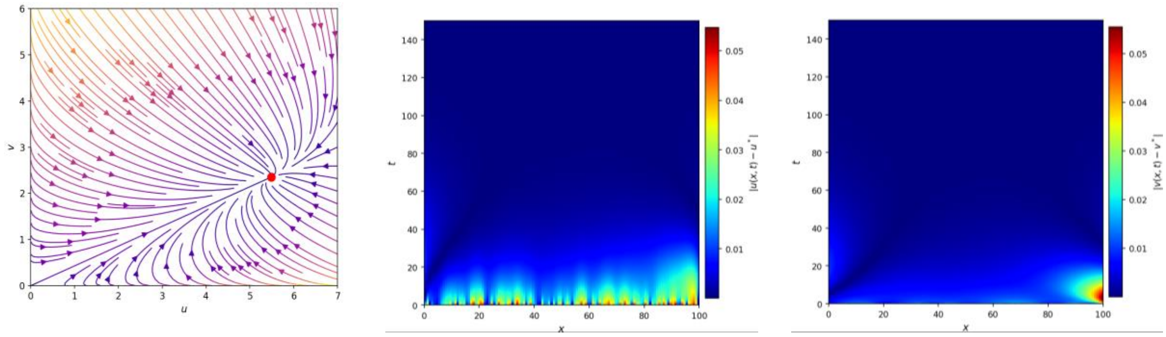
$$u_i^{n+1} = u_i^n + \Delta t!\left[D_u \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + s_K\,(u_i^n)^\alpha (v_i^n)^\beta - \mu\, u_i^n\right],$$

$$v_i^{n+1} = v_i^n + \Delta t!\left[D_v \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{\Delta x^2} + s_H\,(u_i^n)^\alpha (v_i^n)^\beta - \mu\, v_i^n\right].$$

The homogeneous Neumann boundary conditions $u_x(0,t) = u_x(L,t) = 0$ and $v_x(0,t) = v_x(L,t) = 0$ are applied to both domain boundaries. The discretization parameters used are $L = 100$, $N_x = 120$, $\Delta x = L/(N_x - 1)$, $\Delta t = 5 \times 10^{-4}$, and $T = 150$. The initial conditions are set as a small perturbation around the equilibrium.

$$u_0(x) = u^* + \xi\, cos(mx),$$
$$v_0(x) = v^* + \eta\, cos(mx)\,.$$

Under these conditions, the economy does not grow indefinitely but instead converges toward the balanced growth path. Therefore, to understand the dynamical system and its convergence mechanisms, numerical simulations are performed using phase diagrams and the space–time evolution of capital deviations, as shown in Figure 2.



(a) (b) (c)

Figure 2. Simulation of the dynamics of an economic growth model with $\alpha = 0.3$, $\beta = 0.25$, $\mu = 0.13$, $s_K = 0.35$, $s_H = 0.15$, $D_u = 0.01$ and $D_v = 30.0$, (a) phase plane at the steady state (u^*, v^*), (b) Space–time evolution of deviations in physical capital, (c) Space–time evolution of deviations in human capital

Figure 2 (a) shows that all trajectories in the (u, v) phase plane converge to the steady state (u^*, v^*), which is asymptotically stable when $\alpha + \beta < 1$. This indicates that, although regions may start with different initial conditions of capital, decreasing returns to scale ensure that capital accumulation driven by investment cannot increase indefinitely and is ultimately offset by effective capital loss, so that the economy moves toward the balanced growth path represented by the steady state (u^*, v^*) (shown in red). This result is obtained under the assumption of a homogeneous system without diffusion, so the dynamics reflect the internal growth mechanism of the regional economy.

Next, to observe the effect of spatial interaction and diffusion on physical and human capital, nonhomogeneous perturbations are introduced. The diffusion coefficients are chosen to be significantly different, i.e., $D_u = 0.01$ and $D_v = 30.0$, to represent the substantially spatially higher mobility of human capital compared to physical capital. Through this approach, the space–time evolution of deviations from the steady state can be observed, as shown in parts (b) and (c) of Figure 2. The initial spatial perturbation generates transient nonhomogeneous patterns, as indicated by the color scale representing the magnitude of absolute deviations from the steady state, where darker colors correspond to smaller deviations and lighter colors to larger deviations. In part (b), spatial patterns appear in the early phase. However, over time, the intensity of the lighter colors gradually diminishes and becomes uniformly darker, indicating that all regions return to the steady state. In part (c), the patterns are smoother and dampened more quickly. In the early phase, the colors are generally darker, suggesting that the diffusion of human capital acts more effectively in reducing regional disparities and accelerating convergence. In the long–run, persistent spatial patterns are not observed.

Within the framework of Hicks–neutral technology, each region's capital stocks follow adjustment paths shaped by both local accumulation dynamics and spatial distribution. The numerical evidence confirms this theorem, deviations from the balanced growth path are transient, and diffusion acts to smooth out, rather than magnify, spatial inequalities. Overall, the phase plane and space-time evolution plots indicate that no spatial mode grows over time, and the system settles uniformly at a stable state (u^*, v^*) regardless of the initial perturbation.

4. CONCLUSION

This study investigated the stability of the Solow model with Hicks–neutral technology and human capital under capital diffusion. The analysis shows that the model admits a balanced growth path that is locally asymptotically stable under the condition of decreasing returns to scale, i.e., $\alpha + \beta < 1$. The addition of diffusion with Neumann boundary conditions does not alter the stability of the system, even when there are significant differences in diffusion coefficients. For all spatial modes, the associated eigenvalues have negative real parts. Consequently, diffusion does not induce Hopf or Turing bifurcation in the system. This is reinforced by numerical simulations showing that deviations are damped over time. Economically, this confirms that the diffusion of physical and human capital does not generate spatial patterns, but encourages interregional convergence toward a stable long–run steady state. This outcome aligns with Villar-Sepúlveda and Champneys [19], who showed that Turing instability requires an activator-inhibitor structure absent here. Thus, diffusion preserves stability rather than inducing pattern formation in the Solow growth framework under decreasing returns to scale. From a sustainable development perspective, the stability result carries a policy implication, under decreasing returns to scale, opening regions to capital flows does not risk amplifying inequality. Rather, diffusion accelerates the convergence of per-effective-worker capital across locations, providing a theoretical basis for integration-oriented regional development policies. Future studies could expand this framework by including more complex mechanisms, like cross-diffusion effects or different production structures, potentially generating spatial patterns in economic growth models.

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